

STUDY OF SOME FREQUENCY POWER LAWS VALID  
FOR CERTAIN MAGNETIC MATERIALS  
AND OF SOME ANALOGUE LAWS FOR ELASTIC MATERIALS\*

D. A. IORDACHE, V. IORDACHE, ȘT. PUȘCĂ<sup>1</sup>

<sup>1</sup> Physics Department, University "Politehnica" Bucharest  
E-mail: [daniordache2003@yahoo.com](mailto:daniordache2003@yahoo.com)

*Received December 21, 2004*

The accomplished studies pointed out [1] that the Christensen's model [2] of the elastic wave attenuation and dispersion involves (as particular cases) the Zener's model [3], the classical Maxwell's [4] and Kelvin-Voigt's [5] models, and even the Müller's model [6], assuming a frequency power law of the merit factor  $Q$  in rocks.

This work pointed out that the frequency dependence of the viscous friction coefficient of the elastic grains oscillations coincides with the frequency dependence of the same parameter of the magnetic domains walls oscillations, corresponding to the resonance model of the ferrimagnetic materials dispersion [7].

## INTRODUCTION

Starting from the well-known finding that the propagation of acoustic pulses requires previously their decomposition in harmonic components, the present work studies the relations between the main theoretical models of the attenuated waves: the multi-relaxation Christensen's model [2], the mono-relaxation Zener's model [3], the classical models [4, 5] and the frequency power law (Müller's) model [6].

Taking into account some similarities between the Müller's model and some phenomenological models of the magnetic dispersion, the common properties as well as the differences between the features of the oscillations of the grains of some elastic media and those of the magnetic domains of ferrimagnetic materials are studied in detail by this work.

### §1. RELATIONS BETWEEN THE MAIN THEORETICAL DESCRIPTIONS OF THE ATTENUATED WAVES PROPAGATION

One of the main goals of this work was to study the possibilities to reduce – by means of a suitable choice of certain (frequency dependent) effective

\* Paper presented at the 5th International Balkan Workshop on Applied Physics, 5–7 July 2004, Constanța, Romania.

parameters – the multi-relaxation Christensen's description [2] to a mono-relaxation type (Zener's) description and – in following – the Zener's type description to an elementary Maxwell's type description. In this aim, starting from the constitutive equation of the Christensen's media:

$$\sigma_{ij} + \sum_{k=1}^L a_{k,ijmn} \frac{d^k \sigma_{mn}}{dt^k} = \sum_{k=0}^L b_{k,ijmn} \frac{d^k \varepsilon_{mn}}{dt^k} \quad (1)$$

(where  $\sigma_{ij}$  and  $\varepsilon_{mn}$  are the elements of the tensors of stresses and strains, respectively, while  $a_{k,ijmn}$  and  $b_{k,ijmn}$  are coefficients related to the materials properties of the studied medium) and from the Cauchy-Newton's motion equation:

$$\rho \ddot{u}_j = \sum_k \frac{\partial \sigma_{jk}}{\partial x_k}, \quad j, k = 1, 2, 3, \quad (2)$$

one finds that for harmonic time dependencies of stresses and strains:

$$\bar{\sigma}(t) = \bar{\sigma}(0) \cdot e^{i\omega t}, \quad \bar{\varepsilon}(t) = \bar{\varepsilon}(0) \cdot e^{i\omega t}, \quad (3)$$

the complex wave equation in Christensen media:

$$\rho \ddot{\bar{u}} = \bar{E} \cdot \bar{u}'' \quad (4)$$

has – for isotropic media - the expression of a Zener's wave equation:

$$\rho (\ddot{u} + \tau_\varepsilon(\omega) \cdot \ddot{u}) = E(\omega) \cdot (u'' + \tau_\sigma(\omega) \cdot u''), \quad (5)$$

but with frequency dependent parameters:

$$\tau_\varepsilon(\omega) = \frac{1}{\omega} \cdot \frac{\text{Im} \prod_{j=1}^L (1 + i\omega \cdot \tau_{j\varepsilon})}{\text{Re} \prod_{j=1}^L (1 + i\omega \cdot \tau_{j\varepsilon})}, \quad (6)$$

$$\tau_\sigma(\omega) = \frac{1}{\omega} \cdot \frac{\text{Im} \left\{ (1-L) \prod_{j=1}^L (1 + i\omega \cdot \tau_{j\varepsilon}) + \sum_{k(\neq j)}^L \left[ (1 + i \cdot \tau_{j\sigma}) \prod_{j=1}^L (1 + i\omega \cdot \tau_{k\varepsilon}) \right] \right\}}{\text{Re} \left\{ (1-L) \prod_{j=1}^L (1 + i\omega \cdot \tau_{j\varepsilon}) + \sum_{k(\neq j)}^L \left[ (1 + i \cdot \tau_{j\sigma}) \prod_{j=1}^L (1 + i\omega \cdot \tau_{k\varepsilon}) \right] \right\}} \quad (7)$$

and:

$$E(\omega) = M_R \frac{\operatorname{Re} \left\{ (1-L) \prod_{j=1}^L (1+i\omega \cdot \tau_{j\varepsilon}) + \sum_{k(\neq j)}^L \left[ (1+i \cdot \tau_{j\sigma}) \prod_{j=1}^L (1+i\omega \cdot \tau_{k\varepsilon}) \right] \right\}}{\operatorname{Re} \prod_{j=1}^L (1+i\omega \cdot \tau_{j\varepsilon})} \quad (8)$$

where  $\tau_{j\varepsilon}$ ,  $\tau_{j\sigma}$  are the relaxation times corresponding to the  $j$ -th ( $j = 1, L$ ) relaxation, under constant strains and stresses, respectively, while  $M_R$  is the relaxed elasticity (stiffness) modulus.

Starting from the definition of the tangent of mechanical losses:

$$\tan \delta(\omega) = \frac{\operatorname{Im}[\bar{E}(\omega)]}{\operatorname{Re}[\bar{E}(\omega)]}, \quad (9)$$

and introducing the effective parameters:

a) *the unrelaxed elasticity (stiffness) modulus*, as:

$$E_0(\omega) \equiv M_u(\omega) = \frac{|\bar{E}(\omega)|}{\cos \delta}, \quad (10)$$

and: b) *the viscous friction coefficient of the elastic grains oscillations*, as:

$$R(\omega) = \rho\omega \cdot \tan \delta(\omega), \quad (11)$$

one finds that the Christensen-Zener's equations [4, 5] reduce also to the Maxwell's type wave equation:

$$\rho \frac{\partial^2 u}{\partial t^2} + R \frac{\partial u}{\partial t} \equiv \rho \ddot{u} + R \dot{u} = E_0 u''. \quad (12)$$

It was shown (8) that itself the frequency power law Müller's model [5] represents a particular case of the: a) multi-relaxation Christensen's model, for a suitable choice of the number of relaxations  $L$  and of the relaxation times  $\tau_{j\varepsilon}$ ,  $\tau_{j\sigma}$ , b) continuous distribution of relaxation times [9].

One finds so that – despite its similitude to the differential equation of the forced oscillations:

$$\rho \ddot{u} + r \dot{u} + ku = F_0 \cos(\omega t), \quad (13)$$

the attenuated waves differential equation (12) differs because – due to the extinction factor of the attenuated waves integrated equation:

$$u(x, t) = u_0 e^{-sx} \cos(\omega t - kx), \quad (14)$$

the restoring force term  $E_0 u''$  involves also a dissipative component. In fact, the complex wave equation:

$$\rho \ddot{\xi} + R \dot{\xi} = E \cdot \xi'', \quad (12')$$

leads – for harmonic excitations:

$$\xi(t) = \xi_0 \cdot e^{i\omega t}, \quad (15)$$

to the equivalent form:

$$\rho \ddot{\xi} = \bar{E}(\omega) \cdot \xi'', \quad (16)$$

where the complex stiffness (elastic modulus) is:

$$\bar{E} = E' + i \cdot E'' = E \cdot \cos^2 \delta (1 + i \cdot \tan \delta), \quad (17)$$

the tangent of the elastic (mechanical) losses and the corresponding quality factor  $Q_e$  being expressed as:

$$\tan \delta(\omega) = \frac{R(\omega)}{\rho \omega} = \frac{1}{Q_e(\omega)}. \quad (18)$$

After many years when the quality factor of elastic waves (in the seismic frequency band, especially) was considered constant, the works [10–15] converged towards a power-law dependence of  $Q_e$  on frequency:  $Q_e \propto f^\gamma$ , with:  $0 < \gamma < 0.5$ . The synthesis of these obtained experimental results was accomplished by G. Müller, whose work [6] remains the basic one in this field. Later confirmations of the Müller's model were given by Carcione [16], Szabo [17] and Blanch [8].

The above results – obtained mainly for the propagation of the seismic waves in rocks – were extended for another frequency fields and different materials by Choudhury *et al.* [18] (for the ultrasonic attenuation on the microstructure of crystallized glass-ceramics), Botvina *et al.* [19] (different power laws in the non-destructive examinations), etc.

## §2. MAIN PHENOMENOLOGICAL MODELS OF THE MAGNETIC PERMEABILITY DISPERSION

The basic theoretical model [20a] of the magnetic permeability dispersion is rather intricate. Because the electrical conductivity of the ferrimagnetic materials is very weak, the corresponding eddy currents can be neglected, even at microscopic level; for this reason, the theory of the rotational and translational magnetic susceptibility dispersion for such materials [20b–20d] allows considerably simpler descriptions of this phenomenon. Taking into account that all these descriptions (as well as the classical theory of ferromagnetic resonance) lead to some resonance and relaxation phenomena, it results that:

a) the most simple phenomenological model of the (ferrimagnetic) permeability dispersion is that of the **forced oscillations of the magnetization domains walls, under the action of an external harmonic magnetic field:**

$$m\ddot{\xi} + r\dot{\xi} + k\xi = A \cdot H_0 \cdot e^{i\omega t}. \quad (13')$$

Assuming a frequency independent friction coefficient  $r$ , one obtains the frequency dependencies of the complex magnetic susceptibility:

$$\chi' = \frac{\chi_o}{1 - \frac{f^2}{f_o^2} + i \frac{f}{f_{rel}}} = \chi - i \cdot \chi'', \quad (19)$$

as well as of the real magnetic susceptibility and of the tangent of magnetic losses  $\tan \delta_m$ :

$$\chi(f) = \frac{1 - f^2/f_o^2}{\left(1 - \frac{f^2}{f_o^2}\right)^2 + \frac{f^2}{f_{rel}^2}} \cdot \chi_o,$$

$$\tan \delta_m(f) = \frac{\chi''}{\chi} = \frac{f_o^2}{f_o^2 - f^2} \cdot \frac{f}{f_{rel}}, \quad (20)$$

where  $\chi_o$  is the static (at null frequency) magnetic susceptibility, while  $f_o, f_{rel}$  are the resonance frequency and the relaxation frequency, respectively:

$$f_o = \frac{1}{2\pi} \sqrt{\frac{k}{m}}, \quad f_{rel} = \frac{k}{2\pi \cdot r}. \quad (21)$$

Taking into account that the phenomenological model of forced oscillations of the magnetization domains walls does not agree to well – for frequency independent viscous friction coefficients  $r$  – with the experimental results (concerning the frequency dependence of the tangent of magnetic losses, especially), some new theoretical models were proposed:

b) **the Sandy-Green's model [21]**, according to whom the frequency dependence of the complex magnetic susceptibility is described by the function:

$$\chi' = \frac{f_x (f_0 + i \cdot f_{rel})}{(f_0 + i \cdot f_{rel})^2 - f^2}, \quad (22)$$

represented by the straightline:

$$\frac{\text{Re}(1/\chi')}{f_0/f_s} + \frac{\text{Im}(1/\chi')}{f_{rel}/f_s} = 1 \quad (23)$$

in the “inverse” Cole-Cole diagram:  $\text{Im}(1/\chi') = f[\text{Re}(1/\chi')]$ , while  $f_s$  ( $< f_o$ ) is a characteristic frequency specific to the studied magnetic material,

c) **the Mikami’s model [22]**, which combines the typical Cole-Cole correlation [23]:

$$\chi' = \frac{\chi_o}{1 + (if/f_1)^\beta}, \quad (24)$$

with a classical resonance-relaxation description:

$$\chi' = \frac{\chi_{of}}{(1 - f^2/f_0^2) + i \cdot f/f_{rel}} + \frac{\chi_{oC}}{1 + (if/f_1)^\beta}, \quad (25)$$

d) **the Naito’s model [24]**, which utilizes frequency power series in order to describe the frequency dependence of the imaginary part  $\mu''$  of the complex magnetic permeability:

$$\mu''(f) = \mu''_{\max} \cdot \frac{\sum_{n=1}^N a_{2n-1} \left( \frac{f}{f_{\mu''_{\max}}} \right)^{2n-1}}{1 + \sum_{n=1}^N a_{2n} \left( \frac{f}{f_{\mu''_{\max}}} \right)^{2n}}. \quad (26)$$

Taking into account that while the classical forced oscillations model (19) uses 3 parameters  $(\chi_o, f_0, f_{rel})$ , the Mikami’s model [22] uses 6 such parameters  $(\chi_{of}, f_0, f_{rel}, \chi_{oC}, f_1, \text{ and } \beta)$ , and the Naito’s model [24] uses  $2N+2$  such parameters  $(\mu''_{\max}, f_{\mu''_{\max}} \text{ and } a_1, a_2, \dots, a_{2N})$  in order to describe the permeability dispersion, it results that – despite the rather good descriptions of the permeability dispersion ensured by the Mikami’s and the Naito’s model, resp. – it seems that the physical explanation of the experimental data could be provided by a simpler model (using less physical parameters).

Starting from this finding, the numerical analysis of the existing experimental data concerning the permeability dispersion of some industrial spinelic ferrimagnetic materials led us to the assumption that:

c) **the frequency dependence of the friction coefficient of the oscillations of the magnetic domains walls could be described by a frequency power law:**

$$r \propto f^n.$$

It was shown [7], that the corresponding frequency dependencies of the real permeability and of the tangent of magnetic losses are:

$$\chi(f) = \chi_o \frac{f_o^2 - f^2}{f_o^2 \left[ \left(1 - \frac{f^2}{f_o^2}\right)^2 + \left(\frac{f}{f_{rel}}\right)^{2(n+1)} \right]},$$

$$\tan \delta_m = \frac{f_o^2}{f_o^2 - f^2} \cdot \left(\frac{f}{f_{rel}}\right)^{n+1}, \quad (27)$$

which agrees with the experimental data (in the limits of the existing errors).

A qualitative comparison of the predictions of these main theoretical models of the permeability dispersion is given by their plots corresponding to the “inverse” Cole-Cole diagram  $\text{Im}(1/\chi') = f[\text{Re}(1/\chi')]$ , from Fig. 1. For a frequency power law corresponding to the viscous friction coefficient of the magnetization domains walls oscillations, we have:

$$\text{Im} \frac{1}{\chi'} = \frac{1}{\chi_o} \left(\frac{f_o}{f_{rel}}\right)^{n+1} \cdot \left[1 - \chi_o \text{Re} \left(\frac{1}{\chi'}\right)\right]^{\frac{n+1}{2}}, \quad (28)$$

and:

$$\frac{d^2 \left[ \text{Im} \left(\frac{1}{\chi'}\right) \right]}{d \left[ \text{Re} \left(\frac{1}{\chi'}\right) \right]^2} = \chi_o \frac{n^2 - 1}{4} \left(\frac{f_o}{f_{rel}}\right)^{n+1} \cdot \left[1 - \chi_o \text{Re} \left(\frac{1}{\chi'}\right)\right]^{\frac{n-3}{2}}. \quad (29)$$

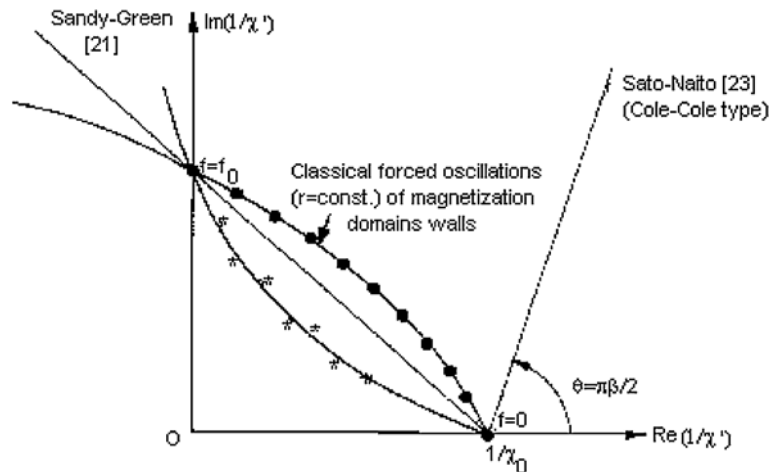


Fig. 1. – Qualitative comparison of the predictions of the main phenomenological models of the permeability dispersion (\* experimental data, in qualitative agreement with the works [6], [19], and in quantitative agreement with [22], [24] and [7] (for  $n > 1$ )).

Taking into account that the plot  $\text{Im}(1/\chi') = f[\text{Re}(1/\chi')]$  corresponding to the experimental data presents a positive curvature, it follows that the phenomenological model [7] agrees with the experimental data for values  $n > 1$  of the frequency power law  $r \propto f^n$ , while the classical model of forced oscillations of magnetization walls ( $n = 0$ ) is in disagreement with the experimental data.

### §3. COMPARISON OF THE FREQUENCY POWER LAWS DESCRIBING DIFFERENT MESOSCOPIC OSCILLATIONS

The detailed study of the mesoscopic oscillations of the: (i) elastic material grains, which lead to the dispersion of the sound waves, and of the: (ii) magnetic domains walls, which lead to the permeability dispersion, points out:

a) their basic features (of the forced oscillations of the elastic grains and of the magnetization domains walls, respectively) are somewhat similar, but they differ because – unlike the restoring force of the magnetic walls which is purely

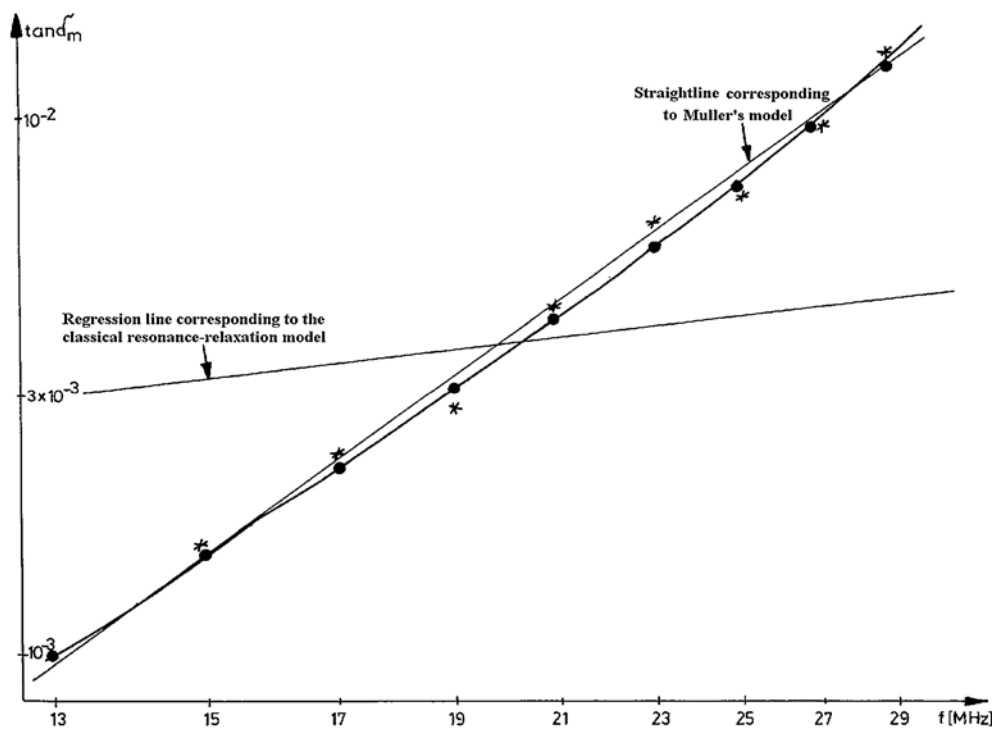


Fig. 2. – Comparison of the phenomenological models corresponding to resonance-relaxation processes (equations (13), (13') above) to the Müller's model [6] and to the frequency dependence of the viscous friction coefficient [7] (curved plot, indicated by the symbols ●) predictions, respectively, relative to the experimental data (indicated by the symbols \*).



elastic, the restoring force corresponding to the attenuated elastic waves involves also a dissipative component,

b) taking into account the relation (18), it results that the (wave) viscous friction coefficient  $R$  obeys also a frequency power law:

$$R = \rho\omega \cdot \tan \delta \propto f^{1-\gamma}; \quad (30)$$

it results so that the “*common denominator*” of the studied mesoscopic oscillations is the frequency power law dependence, both of the elastic wave  $R$  and or the magnetic walls  $r$  viscous friction coefficients;

c) due to the asymmetry of the differential equations of the mesoscopic oscillations of the magnetic domains walls and of the material (medium) grains, resp., the common basic feature: the same frequency power law of the viscous friction coefficient leads to different frequency dependencies of the losses tangents of the: (i) attenuated waves:  $\tan \delta \propto f^{-\gamma}$ , (ii) radio-frequency ferrimagnetic materials (the considerably more intricate frequency dependence given by the equation (27) of our theoretical model [7]). For this reason, the Müller’s model [6] is not compatible relative to the existing experimental data for the permeability dispersion (see Fig. 2). Particularly, the plot  $\ln(\tan \delta) = F(\ln f)$  corresponding to the experimental data presents a positive curvature [25] (p. 269), [26].

## CONCLUSIONS

The main findings of this work are the following:

1) the classical theoretical models of the attenuated elastic waves and of the magnetic permeability dispersion, resp. are somewhat similar, but they differ because the restoring force term  $E \frac{\partial^2 u}{\partial x^2}$  corresponding to the attenuated waves oscillations of the mesoscopic grains involves also a dissipative component,

2) the “*common denominator*” of the models [6] (attenuated elastic waves in rocks, especially) and [7] (magnetic permeability dispersion) consists in the frequency power law dependence of the viscous friction coefficients,

3) it is sufficient to assume a frequency power law dependence of the viscous friction coefficient of the classical phenomenological model (of the forced oscillations of the magnetization domains walls) in order to obtain a compatible model [7] relative to the experimental data concerning the permeability dispersion of the studied spinelic ferrimagnetic materials; this model [7] requires only 4 material parameters, while the other compatible models need 6 material parameters (the Mikami’s model [22]) and  $2N + 2$  ( $N \geq 2$ ) parameters, respectively (the Naito’s model [24]),

4) the published works [26, 27] after paper [7] support the hypothesis of the frequency power law dependence of the viscous friction coefficient of the magnetic domains walls oscillations.

## REFERENCES

1. a) P. P. Delsanto, M. Scalerandi, V. Agostini, D. Iordache, *Il Nuovo Cimento*, **B114**, 1413–1426 (1999); b) D. Iordache, M. Scalerandi, V. Iordache, *Rom. Journ. Phys.*, **45** (9–10), 685–704 (2000).
2. R. B. Christensen, “Theory of Viscoelasticity: an Introduction”, Academic Press, New York, 1982.
3. a) C. Zener, *J. Appl. Phys.*, **18**, 1022 (1947); b) C. Zener, *Physica*, **15** (1-2), 111–118 (1949).
4. N. J. Buckingham, “Acoustic Pulses Propagation in Dispersive Media”, p. 19–34 in “New Perspectives on Problems in Classical and Quantum Physics”, part II, P. P. Delsanto, W. Saenz eds., Gordon and Breach Science Publishers, Amsterdam, 1998.
5. M. Voigt, “Über innere Reibung fester Körper, insbesondere der Metalle”, *Ann. Phys. Chem. Neue Folge*, **47**, 671–693 (1892).
6. G. Müller, “Rheological properties and velocity dispersion of a medium with power-law dependence of  $Q$  on frequency”, *J. Geophys.*, **54**, 20–29 (1983).
7. L. Daniello, D. Iordache, I. M. Popescu, I. Socol, D. Popescu, *Rev. Roum. Phys.*, **25**(2) 193–198 (1980) (submitted in 1978).
8. J. O. Blanch, J. O. A. Robertson, W. W. Symes, *Geophysics*, **60** (1), 176–184 (1995).
9. H. Schäffer, E. Sternin, R. Stannarius, M. Arndt, F. Kremer, *Phys. Rev. Lett.*, **76** (12), 2177 (1996).
10. S. A. Sipkin, T. Jordan, *Bull. Seismol. Soc. Am.*, **69**, 1055–1079 (1979).
11. D. L. Anderson, J. B. Minster, *Geophys. J. R. Astron. Soc.*, **58**, 431–440 (1979).
12. J. B. Minster, “Anelasticity and attenuation”, in “Physics of the Earth’s interior, Proceedings of the Enrico Fermi International School of Physics”, A. Dziewonski, E. Boschi, eds., pp. 152–212, Amsterdam, North-Holland, 1980.
13. G. M. Lundquist, V. F. Cornier, *J. Geophys. Res.*, **85**, 5244–5256 (1980).
14. A. Korpel, *Journ. Acoust. Soc. Am.*, **97**, 1954–1958 (1980).
15. H. Berkhemer, W. Kaufmann, E. Aulbach, H. Schmelling, *Phys. Earth Planet. Inter.*, **29**, 30–41 (1982).
16. a) J. M. Carcione, D. Kosloff, R. Kosloff, *Geophysics*, **93**, 393–407 (1988); b) *ibid.*, *Geophysics*, **95**, 597–611 (1988).
17. T. L. Szabo, *Journ. Acoust. Soc. Am.*, **97**, 14–24 (1995).
18. K. Choudhury, A. K. De, K. K. Phani, *J. Mater. Sci. Lett.*, **13**, 310–312 (1994).
19. L. Botvina, L. Ju. Fradkin, B. Bridge, “Power laws and Generalized Dimensional Analysis in Ultrasonic NDE”, *Nondestructive Testing and Evaluation*, **12**, 103–118 (1995).
20. a) L. D. Landau, E. M. Lifschitz, *Sov. Phys.*, **8**, 153 (1935); b) L. Snoek, *Physica*, **14**, 207 (1948); c) Ch. Kittel, *Phys. Rev.*, **80**, 918 (1950); d) Ch. Kittel, *J. Phys. Rad.*, **12**, 291 (1951).
21. F. Sandy, J. J. Green, *J. Appl. Phys.*, **38** (3), 1413 (1967).
22. I. Mikami, *Jap. J. Appl. Phys.*, **12** (5), 678 (1973).
23. S. Sato, Y. Naito, *Trans. IECE*, **58C**, 421 (1975).
24. a) Y. Naito, *Electronics and Communications in Japan*, **53C**, 148 (1970); b) *ibid.*, **56C**, 118 (1973); c) *ibid.*, *Trans. IECE*, **59C**, 297 (1976).
25. J. Smith, H. P. J. Wijn, “Ferrites”, Wiley, New York, 1959.
26. M. T. Johnson, A. Noodermeer, M. M. E. Severin, W. A. M. Meuwissen, *J. Magnetism and Magnetic Materials*, **116**, 169–176 (1992).
27. L. Burileanu, D. Iordache, L. Daniello, T. Pop, V. Iordache, M. Sărăcin, *Sci. Bull. “Politehnica” University Bucharest, series A*, **56** (1–2), 71–82 (1994).