

VORTEX DYNAMICS IN CAPACITIVELY
COUPLED DOUBLE-LAYERED JOSEPHSON JUNCTION ARRAYS:
HALL STATES AND DRAG EFFECT

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I analyze the transport coefficients in bilayer Josephson Junction arrays in the presence of frustration due to applied magnetic fields and offset charges. By using a duality transformation to map the action of the vortex-array-system on to the action for a bilayer quantum Hall system, I find that the Hall conductivity is quantized and that the system shows an enhanced counterflow drag.

Two-dimensional arrays of ultra-small superconducting grains, weakly coupled by Josephson junctions, are known to provide rich quantum dynamics at the macroscopic level [1]. More interesting dynamics comes to play when frustration is introduced by magnetic fields [2] and by external charges [3]; these control the numbers of vortices and Cooper pairs.

Two characteristic energy scales characterize these systems: E_J the Josephson energy is associated with Cooper pairs tunneling between islands, and E_c the charging energy needed to add extra charges to an island. The competition between electron effects and Josephson effect gives rise to a wealth of classical ($E_c/E_J \ll 1$) and quantum ($E_c/E_J \gg 1$) phenomena.

We consider two identical parallel Josephson Junction arrays with capacitive coupling between them. This system has been investigated in [4, 5] with the aim of showing the existence of a duality between charges and vortices. The focus in [4] was on the situation when one array is in the quasi-classical (vortex) regime while the other is in the quantum (charge) regime. The resulting effective action describes dual charges in one array and vortices in the other, and in contrast to the one-layer problem, these are dynamic degrees of freedom. Another appealing feature of coupling two layers of Josephson junction arrays is the possibility to tune independently the interlayer capacitance and thereby control the interaction between vortices and charges.

Our study is motivated by a remarkable analogy between vortex dynamics and cyclotronic motion of electrons [6]. In fact, vortices accumulate geometric

phases when transported around closed paths [7]. These phases are 2π times the number of bosons encircled, implying that vortices see each boson effectively attached to one unit of a fictitious magnetic flux. Together with the more familiar Aharonov-Bohm phase acquired by charged particles when transported around a magnetic flux, this leads to the concept of charge-vortex duality—charge density is represented as the curl of a fictitious magnetic field. The existence of this Berry phase allows fractional statistics for a system of vortices. This has some similarities with Fractional quantum Hall systems involving two-dimensional electrons [8]. The latter exhibit very rich properties when the filling factor (ratio between the electron density and the density of flux quanta) is varied, and more interesting effects arise when multilayer coupling is added—the layer index becomes a new degree of freedom [9]. For electronic layers, through studies of the transresistance (Drag resistance), one can extract information not only about the direct Coulomb interaction between electrons in different systems, but also about the collective modes of the coupled systems [10].

The aim of the present work is to analyze the dynamics of vortices in each array (intra-layer interaction) and their influence on vortices residing in the other array (inter-array interaction), and to explore the formation of incompressible quantum Hall states and drag effect of vortices between layers.

The Hamiltonian for capacitively coupled double-layered Josephson Junction arrays is

$$\hat{H} = \frac{4e^2}{2} \sum_{\langle rr' \rangle} (n_r^{(\alpha)} - \bar{n}_x) [C^{-1}]_{\alpha\alpha'}^{rr'} (n_{r'}^{(\alpha')} - \bar{n}_x) - E_J \sum_{\langle rr' \rangle} \cos(\phi_r^{(\alpha)} - \phi_{r'}^{(\alpha')} - A_{rr'}^{(\alpha)}) \quad (1)$$

where $n_r^{(\alpha)}$ and $\phi_r^{(\alpha)}$ denote respectively the charge and the phase of the superconducting order parameter of the r -th island in each array ($\alpha = 1, 2$), and the sum is over nearest neighbors. The competition between these two canonically conjugated variables is captured by the usual commutation relation $[n_r^{(\alpha)}, \phi_{r'}^{(\alpha')}] = i\delta_{rr'}\delta^{\alpha\alpha'}$. We assume that each array is in a perpendicular magnetic field B to induce vortices in the system, and we allow for offset charges on each superconducting grain $Q_x = 2e\bar{n}_x$; these can be controlled by external gate voltages between the islands and the ground. The Josephson coupling constant, E_J , is assumed to be identical in each array. The inverse matrix describing the interaction between charges is given in Fourier representation as

$$\hat{C}^{-1} = \frac{1}{[C_I + Cq^2]^2 - C_I^2} \begin{pmatrix} C_I + Cq^2 & C_I \\ C_I & C_I + Cq^2 \end{pmatrix} \quad (2)$$

where C is the capacitance of the junction assumed to be identical in each array, and C_I is the interlayer capacitance between each island in one array coupled parallel to one island (straight coupling) in the other array. Note that the interlayer capacitance C_I not only couples the layers, but also introduces a self-capacitance of each island to the ground.

To achieve a formulation in terms of vortices, the action of the system is obtained by analyzing the partition function $Z = \text{Tr}_\phi e^{-\beta \hat{H}}$. This is performed stepwise and its derivation follows closely previous work [11]. Using a Villain approximation and a duality transformation, the partition function is transformed into a path integral over integer fields $J_\mu^{(\alpha)}$, $\mu = 0, 1, 2$ describing vortex currents in each layer coupled to two real gauge fields $a_\mu^{(\alpha)}$, describing the charge degrees of freedom. The field strengths associated with these gauge fields, $\frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\mu a_\nu^{(\alpha)}$, are Cooper pairs currents and density on each site of the array. In a gauge in which $\partial_i \cdot a_i^{(\alpha)} = 0$, the action is found to be

$$S = \sum_r i(\rho_r^\alpha - \bar{\rho}) a_{0r}^\alpha + i\vec{J}_r^\alpha \cdot (\vec{a}_r^\alpha + \vec{a}_c) + \frac{1}{8\pi^2 E_J} [(\nabla_i \vec{a}_r^\alpha)^2 + (\vec{\nabla} a_{0r}^\alpha)^2] + \frac{e^2}{2\pi^2} \sum_{rr'} (\vec{\nabla} \wedge \vec{a}_r^\alpha) [C^{-1}]_{\alpha\alpha'}^{rr'} (\vec{\nabla} \wedge \vec{a}_{r'}^{\alpha'}) \quad (3)$$

where $\vec{\nabla} \wedge \vec{a}_c = 2\pi \vec{n}_x$. Mutual interactions between vortices and their self-interactions are mediated through exchanges of a_μ^α . This can be seen easily in the continuum limit by integrating out $a_0^{(\alpha)}$, yielding a logarithmic density-density interaction between vortices

$$\pi E_J \int d\vec{r} \int d\vec{r}' [\rho^\alpha(r) - \bar{\rho}] \ln|r - r'| [\rho^{\alpha'}(r') - \bar{\rho}] \quad (4)$$

Similarly, integrating out $\vec{a}^{(\alpha)}$ yields inter-layer and intra-layer current-current interactions between vortices. In momentum space and for frequencies $\omega \ll \omega_p = \sqrt{8E_J E_C}$, this is given by

$$\pi^2 \int d\omega \int d\vec{q} \vec{J}_1 \left[\frac{C}{4e^2} + \frac{E_J}{\omega^2 + v_s^2 q^2} \right] \vec{J}_1 + \vec{J}_2 \left[\frac{C}{4e^2} + \frac{E_J}{\omega^2 + v_s^2 q^2} \right] \vec{J}_2 + \vec{J}_1 \left[\frac{C}{4e^2} - \frac{E_J}{\omega^2 + v_s^2 q^2} \right] \vec{J}_2 \quad (5)$$

Some comments about the effect and nature of the current-current interaction are in order. First, as seen from the previous equation a mass term associated to the vortices is induced ($m_{bare} = \frac{\pi^2 C}{2e^2}$), this depends only on the junction capacitance. For typical arrays with $C = 1$ fF, the vortex mass is 500 times smaller than the electron mass. This small value indicates that quantum effects are likely to occur. Second, the coupling between layers ($v_s^2 = 2e^2 E_J / C_I$) makes this interaction non local in time, and therefore introduces a mechanism of vortex dissipation [12]. Further to the above mentioned interaction, vortices are also subject to a periodic lattice potential. In the continuum model considered here, we take account of this potential by promoting the bare mass to a band mass, m_{band} [13]. The vortices dynamics then resembles that of interacting bosons of mass m_{band} and average density $\bar{\rho}$, under the effect of a fictitious magnetic field $\bar{\nabla} \wedge \bar{a}_c = 2\pi \bar{n}_x$.

To further analyze the properties of this system we adopt now a Landau-Ginzburg description in terms of two bosonic fields $\psi^{(\alpha)}$ that play the role of order parameters of vortices in each layer. The dynamics of these fields is governed by the following action

$$\begin{aligned}
S^{vor} = & \int d\tau \int d\vec{r} \psi^{\alpha*} \nabla_\tau \psi^\alpha + \frac{1}{2M} \left| (i\bar{\nabla} - \bar{a}^\alpha - \bar{a}_c) \psi^\alpha \right|^2 + \\
& + \int d\tau \int d\vec{r} \int d\vec{r}' \pi E_J \left[|\psi^\alpha|^2 - \bar{\rho} \right] \ln |r - r'| \left[|\psi^{\alpha'}|^2 - \bar{\rho} \right] + \\
& + \int d\tau \int d\vec{r} \frac{1}{8\pi^2 E_J} (\nabla_\tau \bar{a}^\alpha)^2 + \\
& + \int d\tau \int d\vec{r} \int d\vec{r}' \frac{e^2}{2\pi^2} (\bar{\nabla} \wedge \bar{a}^\alpha(r)) C_{\alpha\alpha'}^{-1}(r-r') (\bar{\nabla} \wedge \bar{a}^{\alpha'}(r'))
\end{aligned} \tag{6}$$

here M is an effective mass such that $m_{band} = M + m_{bare}$.

A similar analysis of vortices in a single Josephson Junction array was carried out in [13], where it was further shown that a Chern-Simons singular gauge transformation that attached an even number of fictitious Cooper pairs to each vortex resulted in an action that was convenient for a saddle point analysis and led to a quantum Hall fluid state. Here we extend the same analysis to two coupled arrays. We perform a boson-boson transformation by using one statistical gauge field $b_\mu = (b_\mu^{(1)} + b_\mu^{(2)})/2$:

$$\psi_\alpha(r) = \exp \left[-i \int dr' b(r') \right] \Phi_\alpha \tag{7}$$

with the constraint $\bar{\nabla} \wedge \bar{b}(r) = k \sum \Phi_\alpha^* \Phi_\alpha$. The combination $(b_1 - b_2)$ decouples

from the system [14]. Writing the boson fields in terms of phases and amplitudes $\Phi_\alpha = \sqrt{\rho_\alpha} \exp(i\theta_\alpha)$, the transformed action reads

$$\begin{aligned}
S^{vor} = & \int d\tau \int d\vec{r} \int d\vec{r}' i\rho^\alpha (\nabla_\tau \theta^\alpha - b_0) + \frac{\rho^\alpha}{2M} (\bar{\nabla} \theta^\alpha - \vec{a}^\alpha - \vec{a}_c + \vec{b})^2 + \\
& + \int d\tau \int d\vec{r} \frac{1}{2M} (\nabla \sqrt{\rho^{(\alpha)}})^2 + \\
& + \int d\tau \int d\vec{r} \int d\vec{r}' \pi E_J \left[|\Psi^{(\alpha)}|^2 - \bar{\rho} \right] \ln |r - r'| \left[|\Psi^{(\alpha')}|^2 - \bar{\rho} \right] + \\
& + \int d\tau \int d\vec{r} \int d\vec{r}' \frac{e^2}{2\pi^2} (\bar{\nabla} \wedge \vec{a}_\alpha(\vec{r})) C_{\alpha\alpha'}^{-1} (\bar{\nabla} \wedge \vec{a}_{\alpha'}(\vec{r}')) + \\
& + \int d\tau \int d\vec{r} \frac{1}{8\pi^2 E_J} (\nabla_\tau \vec{a}_\alpha)^2 + \frac{i}{4\pi k} \int d\tau \int d\vec{r} \varepsilon^{\mu\nu\lambda} b_\mu \nabla_\nu b_\lambda
\end{aligned} \tag{8}$$

The advantage of such a transformation is that it allows for a mean-field solution given by

$$\langle \rho_1 + \rho_2 \rangle = \bar{\rho}_{tot}, \tag{9}$$

$$\langle \bar{\nabla} \theta_1 \rangle = \langle \bar{\nabla} \theta_2 \rangle = 0 \tag{10}$$

$$\langle \vec{b} \rangle = \vec{a}_c \tag{11}$$

$$\langle \vec{a}_1 \rangle = \langle \vec{a}_2 \rangle = 0 \tag{12}$$

In this state the total vortex density $\rho_1 + \rho_2$ is constant on the average and the Chern-Simons gauge field \vec{b} cancels the effective field \vec{a}_c associated with Cooper pairs offset charges. Furthermore, the condensation of vortices in this state is unaffected by small variations of the density of vortices in each layer as long as the total density $\rho_1 + \rho_2$ is fixed. This results in a gapless mode associated with fluctuations in the difference density $\rho_1 - \rho_2$.

Furthermore, the solution satisfies all equations of motion, and when combined with the equation of motion of b_0 , it yields the condition $\bar{\nabla} \wedge \vec{a}_c = 2\pi k \bar{\rho}_{tot}$. Since the effective magnetic density (Cooper pairs density) is given by $2\pi \bar{n}_x$, one can introduce a filling factor $\nu = \bar{\rho}_{tot} / \bar{n}_x = 1/k$. In this case the transformed bosons (composites of vortices and effective flux tubes) see no net field on the average and form a Bose-Einstein condensate. The Meissner effect of this charged Bose condensate leads to the incompressibility of the original vortex problem. To see this we examine the fluctuations about the mean field solution by writing $\delta \rho_\alpha = \rho_\alpha - \bar{\rho}$, $\delta \vec{b} = \vec{b} - \vec{a}_c$.

To calculate transport coefficients, we couple the vortices in each array to probing gauge fields $A_\mu^{(\alpha)}$ and we integrate out all Gaussian fluctuations in the bosonic fields ρ_α , θ_α and gauge fields a_μ^α and b_μ . In doing that, it was convenient to combine all fields into symmetric and antisymmetric combinations, $\rho^\pm = (\rho^{(1)} \pm \rho^{(2)})/\sqrt{2}$, $\bar{a}^\pm = (\bar{a}_1 \pm \bar{a}_2)/\sqrt{2}$, $A_\mu^\pm = (A_\mu^{(1)} \pm A_\mu^{(2)})/\sqrt{2}$, and make use of the constraint equation obtained from Eq. (8) by varying with respect to b_0 : $\rho^\pm = \frac{1}{2\pi k\sqrt{2}} \bar{\nabla} \wedge \bar{b}$. The result of such derivation in terms of longitudinal and transverse fields A_0^\pm , A_\perp^\pm is

$$S_{eff} [A_0^\pm, A_\perp^\pm] = -\frac{1}{2} \int A_0^\pm(-q) \Sigma_{00}^\pm(q) A_0^\pm(q) + A_0^\pm(-q) \Sigma_{0\perp}^\pm(q) A_\perp^\pm(q) + A_\perp^\pm(-q) \Sigma_{\perp\perp}^\pm(q) A_\perp^\pm(q). \quad (13)$$

Here $\hat{\Sigma}^\pm$ is a 2×2 matrix defined such that $J_{vor}^\pm = \hat{\Sigma}^\pm A_\perp^\pm$, and whose inverse has elements given by:

$$-\left(\frac{1}{\hat{\Sigma}^\pm}\right)_{00} = \frac{M\omega^2}{\bar{\rho}q^2} + \frac{q^2}{4M\bar{\rho}} + \frac{4\pi^2 E_J}{q^2} \quad (14)$$

$$-\left(\frac{1}{\hat{\Sigma}^\pm}\right)_{\perp\perp} = \frac{M}{\bar{\rho}} + \frac{\pi^2}{\omega^2/4E_J + q^2 e^2 (C_{11}^{-1} \pm C_{12}^{-1})} \quad (15)$$

$$\left(\frac{1}{\hat{\Sigma}^+}\right)_{0\perp} = \frac{4\pi i k}{q} \quad (16)$$

$$\left(\frac{1}{\hat{\Sigma}^-}\right)_{0\perp} = 0 \quad (17)$$

This action encodes all information about the phenomenology of vortex dynamics. For instance the compressibility of the vortex system is found from the density-density correlation function:

$$\frac{\delta^2 S_{eff}}{\delta A_0^\alpha(-q) \delta A_0^\alpha(q)} = \frac{q^2 \bar{\rho}/M}{\omega^2 + \omega_q^2} \quad (18)$$

where $\omega_q^2 = 16\pi^2 k^2 (\bar{\rho}/M)^2 [1 + \bar{\rho}\pi^2/2ME_C] + 4\pi^2 E_J (\bar{\rho}/M) + \frac{q^4}{4M^2}$. The long wavelength limit of this correlation function gives a zero compressibility. Central to this result is the finiteness of the vortex Bose density. On the other

hand, the symmetric and antisymmetric vortex currents correlation functions display, in the long wavelength limit, two completely different behaviors:

$$\langle J_{\perp}^{+}(-q)J_{\perp}^{+}(q) \rangle \sim \frac{1}{M/\rho + \pi^2 C/e^2 + k^2/E_J} \quad (19)$$

$$\langle J_{\perp}^{-}(-q)J_{\perp}^{-}(q) \rangle \sim (\omega^2 + v_s^2 q^2)/4\pi^2 E_J \quad (20)$$

The former correlation function is constant in the $\omega \rightarrow 0$ limit and is characteristic of superconductors; this is due to the long range logarithmic interaction. While the latter correlation function displays an insulator-like behavior; this results from the interlayer capacitive coupling between the arrays that introduces dissipation.

Next we examine the conductivity matrix relating the vortex current to the driving total gauge fields that include the probing fields and the induced fields by the vortices. $J_{\mu} = \sigma_{\mu\nu} E_{\nu}^{total}$ where $E_{\parallel}^{total}(q) = -\frac{iq}{2\pi} A_0^{total}(q)$ and $E_{\perp}^{total}(q) = -\frac{i\omega}{2\pi} A_{\perp}^{total}(q)$.

From the action Eq. (8), the induced gauges fields are given by $A_{\mu}^{ind} = \hat{V}_{\mu\nu} J_{\nu}$ where

$$\hat{V}^{\pm} = \begin{pmatrix} \frac{4\pi^2 E_J}{q^2} & 0 \\ 0 & \frac{\pi^2}{\omega^2/4E_J + q^2 e^2 (C_{11}^{-1}(q) \pm C_{12}^{-1}(q))} \end{pmatrix} \quad (21)$$

and the vortex conductivity matrix is given by

$$\sigma_{\mu\nu}^{\pm} = \begin{pmatrix} -\frac{\omega}{q} & 0 \\ 0 & 1 \end{pmatrix} (\hat{\Sigma}_{\pm}^{-1} + \hat{V}_{\pm})^{-1} \begin{pmatrix} \frac{2\pi i}{q} & 0 \\ 0 & \frac{2\pi i}{\omega} \end{pmatrix} \quad (22)$$

From this we find to leading order in q , ω , the symmetric (bilayer), σ^{+} , and antisymmetric (counterflow), σ^{-} , conductivities:

$$\sigma_{0\perp}^{+} = \frac{1}{4k} \quad (23)$$

$$\sigma_{0\perp}^{-} = 0 \quad (24)$$

$$\sigma_{00}^{+} = -\sigma_{\perp\perp}^{+} = \frac{iM\omega}{8\pi\bar{\rho}k^2} \quad (25)$$

$$\sigma_{00}^- = \sigma_{\perp\perp}^- = \frac{-i2\pi\bar{\rho}}{M\omega} \quad (26)$$

These results show that $\sigma_{\perp\perp}^+$ decreases with decreasing frequency, while the longitudinal counterflow conductivity $\sigma_{\perp\perp}^-$ diverges as the frequency is reduced to zero and the Hall counterflow vanishes. This indicates that the counterflow current is transported by neutral carriers, that is, pairs of vortices and anti-vortices, which are bound by the electrostatic energy coupling capacitance C_f . The total vortex current is perpendicular to the total driving force

$$\vec{J}^+ = \frac{i}{8\pi k} \hat{z} \times (\omega \vec{A}^+ + \vec{q} A_0^+) \quad (27)$$

and the difference vortex current satisfies

$$\vec{J}^- = -\frac{\bar{\rho}}{M\omega} (\omega \vec{A}^- + \vec{q} A_0^-) \quad (28)$$

As a consequence the Hall conductivities in both layers are quantized $\sigma_{xy}^{11} = \sigma_{xy}^{22} = 1/8k$ and the drag Hall conductivity is also quantized $\sigma_{xy}^{12} = 1/8k$, namely, a Hall driving force vector in one layer induces a dragged current in the second layer. Furthermore, in the longitudinal direction the system exhibits a perfect drag and carries vortex currents along the two arrays equally large in magnitude but opposite in direction $\sigma_{xx}^{11}(\omega) = -\sigma_{xx}^{12}(\omega) = \pi\bar{\rho}/(mi\omega)$.

To summarize, I have investigated the dynamics of vortices in two identical parallel Josephson Junction arrays with capacitive coupling between them, in the presence of frustration due to applied magnetic fields and induced external charges. I have shown that this dynamics resembles that of massive interacting charged particles under the effect of an effective magnetic field and gauge fields. Using a Chern-Simons-Ginzburg-Landau approach, I have examined some of the properties of the resulting quantum Hall state at appropriate values of the filling factor, which is controlled by the gate voltage as well as the magnetic field. It is found that the vortices conductivity shows a quantized Hall conductivity in each layer separately and also a quantized Hall drag conductivity between the two layers. Furthermore, the longitudinal vortex conductivity shows an enhanced drag effect resulting into two mirror currents. This counterflow current involves pairs of vortices and anti-vortices, which are bound by the electrostatic energy coupling capacitance C_f . Whence this effect is robust for strong coupling between the layers.

Experimentally, the drag effect should be achieved in principal and a recent experiment [15] has already measured the transport properties of capacitively coupled one-dimensional arrays of small Josephson junctions. It was found that the samples act as current mirrors; the currents in the two arrays couple to each

other and become equally large. We expect that under suitable biasing conditions the same results can be achieved for two-dimensional arrays.

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