

DECOHERENCE IN OPEN QUANTUM SYSTEMS

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In the framework of the Lindblad theory for open quantum systems we determine the degree of quantum decoherence of a harmonic oscillator interacting with a thermal bath.

1. INTRODUCTION

The transition from quantum to classical physics and classicality of quantum systems continue to be among the most interesting problems in many fields of physics, for both conceptual and experimental reasons [1, 2, 3]. Two conditions are essential for the classicality of a quantum system [4, 5]: a) quantum decoherence (QD), that means the irreversible, uncontrollable and persistent formation of quantum correlation (entanglement) of the system with its environment [6], expressed by the damping of the coherences present in the quantum state of the system, when the off-diagonal elements of the density matrix of the system decay below a certain level, so that this density matrix becomes approximately diagonal and b) classical correlations, expressed by the fact that the Wigner function of the quantum system has a peak which follows the classical equations of motion in phase space, that is the quantum state becomes peaked along a classical trajectory.

In the last two decades it has become more and more clear that the classicality is an emergent property of open quantum systems, since both main features of this process – quantum decoherence and classical correlations – strongly depend on the interaction between the system and its external environment [7]. A remarkable aspect of the current research helping in understanding the nature of the quantum to classical transition is that for the first time there have recently been carried on experiments probing the boundary between the quantum and the classical domains in a controlled way [3].

In most of literature, the quantum decoherence has been studied for a system coupled to an environment or thermal bath with many degrees of freedom. The main purpose of this paper is to study QD for a harmonic oscillator

interacting with an environment in the framework of the Lindblad theory for open quantum systems. More concretely we determine the degree of QD for a system consisting of a harmonic oscillator in a thermal bath. For that purpose, we find the evolution of the density matrix and of the Wigner function of the considered system and then we apply the criterion of QD. It is found that the system manifests a QD which is more and more significant in time.

The organizing of the paper is as follows. In Sec. 2 we write the Lindblad master equation for the damped harmonic oscillator and in Sec. 3 we derive the master equation in coordinate representation and the corresponding Fokker-Planck equation in the Wigner representation and find the density matrix and Wigner function of the considered system. Then in Sec. 4 we investigate QD and analyze it quantitatively. A summary and concluding remarks are given in Sec. 5.

2. LINDBLAD MASTER EQUATION FOR THE HARMONIC OSCILLATOR

Here we review the Lindblad's axiomatic formalism based on quantum dynamical semigroups. The irreversible time evolution of an open system is described by the following general quantum Markovian master equation for the density operator $\rho(t)$ [8, 9, 10]:

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] + \frac{1}{2\hbar} \sum_j ([V_j \rho(t), V_j^\dagger] + [V_j, \rho(t) V_j^\dagger]). \quad (1)$$

H is the Hamiltonian operator of the system and V_j, V_j^\dagger are operators on the Hilbert space of the Hamiltonian, which model the environment. In order to obtain, for the damped quantum harmonic oscillator, equations of motion as close as possible to the classical ones, the two possible operators V_1 and V_2 are taken as linear polynomials in coordinate q and momentum p [11, 12, 13] and the harmonic oscillator Hamiltonian H is chosen of the general quadratic form

$$H = H_0 + \frac{\mu}{2}(qp + pq), \quad H_0 = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2. \quad (2)$$

With these choices the master equation (1) takes the following form [12, 13]:

$$\begin{aligned} \frac{d\rho}{dt} = & -\frac{i}{\hbar}[H_0, \rho] - \frac{i}{2\hbar}(\lambda + \mu)[q, \rho p + p\rho] + \frac{i}{2\hbar}(\lambda - \mu)[p, \rho q + q\rho] \\ & - \frac{D_{pp}}{\hbar^2}[q, [q, \rho]] - \frac{D_{qq}}{\hbar^2}[p, [p, \rho]] + \frac{D_{pq}}{\hbar^2}([q, [p, \rho]] + [p, [q, \rho]]). \end{aligned} \quad (3)$$

The quantum diffusion coefficients D_{pp} , D_{qq} , D_{pq} and the dissipation constant λ satisfy the following fundamental constraints [12, 13]: $D_{pp} > 0$, $D_{qq} > 0$ and

$$D_{pp}D_{qq} - D_{pq}^2 \geq \frac{\lambda^2 \hbar^2}{4}. \quad (4)$$

In the particular case when the asymptotic state is a Gibbs state $\rho_G(\infty) = e^{-\frac{H_0}{kT}} / \text{Tr} e^{-\frac{H_0}{kT}}$, these coefficients become [12, 13]

$$D_{pp} = \frac{\lambda + \mu}{2} \hbar m \omega \coth \frac{\hbar \omega}{2kT}, \quad D_{qq} = \frac{\lambda - \mu}{2} \frac{\hbar}{m \omega} \coth \frac{\hbar \omega}{2kT}, \quad D_{pq} = 0, \quad (5)$$

where T is the temperature of the thermal bath. In this case, the fundamental constraints are satisfied only if $\lambda > \mu$ and

$$(\lambda^2 - \mu^2) \coth^2 \frac{\hbar \omega}{2kT} \geq \lambda^2. \quad (6)$$

Lindblad has proven [11] that in the Markovian regime the harmonic oscillator master equation which satisfies the complete positivity condition cannot satisfy simultaneously the translational invariance and the detailed balance (which assures an asymptotic approach to the canonical thermal equilibrium state). The necessary and sufficient condition for translational invariance is $\lambda = \mu$ [11, 12, 13]. If $\lambda \neq \mu$, then we violate translational invariance, but we keep the canonical equilibrium state.

The relation (4) is a necessary condition for the generalized uncertainty inequality

$$\sigma_{qq}(t)\sigma_{pp}(t) - \sigma_{pq}^2(t) \geq \frac{\hbar^2}{4} \quad (7)$$

to be fulfilled, where σ_{qq} and σ_{pp} denote the dispersion (variance) of the coordinate and momentum, respectively, and σ_{pq} denotes the correlation (covariance) of the coordinate and momentum. The equality in relation (7) is realized for a special class of pure states, called correlated coherent states [14] or squeezed coherent states.

The asymptotic values $\sigma_{qq}(\infty)$, $\sigma_{pp}(\infty)$, $\sigma_{pq}(\infty)$ do not depend on the initial values $\sigma_{qq}(0)$, $\sigma_{pp}(0)$, $\sigma_{pq}(0)$ and in the case of a thermal bath with coefficients (5), they reduce to [12, 13]

$$\sigma_{qq}(\infty) = \frac{\hbar}{2m\omega} \coth \frac{\hbar \omega}{2kT}, \quad \sigma_{pp}(\infty) = \frac{\hbar m \omega}{2} \coth \frac{\hbar \omega}{2kT}, \quad \sigma_{pq}(\infty) = 0. \quad (8)$$

3. DENSITY MATRIX AND WIGNER DISTRIBUTION FUNCTION

We consider a harmonic oscillator with an initial Gaussian wave function

$$\Psi(q) = \left(\frac{1}{2\pi\sigma_{qq}(0)} \right)^{\frac{1}{4}} \times \exp \left[-\frac{1}{4\sigma_{qq}(0)} \left(1 - \frac{2i}{\hbar} \sigma_{pq}(0) \right) (q - \sigma_q(0))^2 + \frac{i}{\hbar} \sigma_p(0) q \right], \quad (9)$$

where $\sigma_{qq}(0)$ is the initial spread, $\sigma_{pq}(0)$ the initial covariance, and $\sigma_q(0)$ and $\sigma_p(0)$ are the initial averaged position and momentum of the wave packet. The initial state (9) represents a correlated coherent state [14] with the variances and covariance of coordinate and momentum

$$\sigma_{qq}(0) = \frac{\hbar\delta}{2m\omega}, \quad \sigma_{pp}(0) = \frac{\hbar m\omega}{2\delta(1-r^2)}, \quad \sigma_{pq}(0) = \frac{\hbar r}{2\sqrt{1-r^2}}. \quad (10)$$

Here, δ is the squeezing parameter which measures the spread in the initial Gaussian packet and r , $|r| < 1$ is the correlation coefficient at time $t = 0$. The initial values (10) correspond to a minimum uncertainty state, since they fulfil the generalized uncertainty relation

$$\sigma_{qq}(0)\sigma_{pp}(0) - \sigma_{pq}^2(0) = \frac{\hbar^2}{4}. \quad (11)$$

For $\delta = 1$ and $r = 0$ the correlated coherent state becomes a Glauber coherent state. For a given temperature T of the bath and for any parameters δ and r the inequality (6) alone determines the range of values of the parameters λ and μ [15].

From Eq. (3) we derive the evolution equation in coordinate representation:

$$\begin{aligned} \frac{\partial \rho}{\partial t} = & \frac{i\hbar}{2m} \left(\frac{\partial^2}{\partial q^2} - \frac{\partial^2}{\partial q'^2} \right) \rho - \frac{im\omega^2}{2\hbar} (q^2 - q'^2) \rho - \\ & - \frac{1}{2} (\lambda + \mu) (q - q') \left(\frac{\partial}{\partial q} - \frac{\partial}{\partial q'} \right) \rho + \frac{1}{2} (\lambda - \mu) \left[(q + q') \left(\frac{\partial}{\partial q} + \frac{\partial}{\partial q'} \right) + 2 \right] \rho \\ & - \frac{D_{pp}}{\hbar^2} (q - q')^2 \rho + D_{qq} \left(\frac{\partial}{\partial q} + \frac{\partial}{\partial q'} \right)^2 \rho - 2iD_{pq} \hbar (q - q') \left(\frac{\partial}{\partial q} + \frac{\partial}{\partial q'} \right) \rho \end{aligned} \quad (12)$$

and in Refs. [16, 17, 18] we transformed the master equation (3) for the density operator into the following Fokker-Planck-type equation satisfied by the Wigner distribution function $W(q, p, t)$:

$$\begin{aligned} \frac{\partial W}{\partial t} = & -\frac{p}{m} \frac{\partial W}{\partial q} + m\omega^2 q \frac{\partial W}{\partial p} + (\lambda + \mu) \frac{\partial}{\partial p} (pW) + (\lambda - \mu) \frac{\partial}{\partial q} (qW) + \\ & + D_{pp} \frac{\partial^2 W}{\partial p^2} + D_{qq} \frac{\partial^2 W}{\partial q^2} + 2D_{pq} \frac{\partial^2 W}{\partial p \partial q}. \end{aligned} \quad (13)$$

The first two terms on the right-hand side of both these equations generate a purely unitary evolution. They give the usual Liouvillian evolution. The third and fourth terms are the dissipative terms and have a damping effect (exchange of energy with environment). The last three are noise (diffusive) terms and produce fluctuation effects in the evolution of the system. D_{pp} promotes diffusion in momentum and generates decoherence in coordinate q : it reduces the off-diagonal terms, responsible for correlation between spatially separated pieces of the wave packet. Similarly D_{qq} promotes diffusion in coordinate and generates decoherence in momentum p . The D_{pq} term is the so-called ‘‘anomalous diffusion’’ term. It promotes diffusion in the variable $qp + pq$, just like both the other diffusion terms, but it does not generate decoherence.

The density matrix solution of Eq. (12) has the general form of Gaussian density matrices

$$\begin{aligned} \langle q | \rho(t) | q' \rangle = & \left(\frac{1}{2\pi\sigma_{qq}(t)} \right)^{\frac{1}{2}} \times \\ & \times \exp \left[-\frac{1}{2\sigma_{qq}(t)} \left(\frac{q+q'}{2} - \sigma_q(t) \right)^2 - \frac{\sigma(t)}{2\hbar^2\sigma_{qq}(t)} (q-q')^2 + \right. \\ & \left. + \frac{i\sigma_{pq}(t)}{\hbar\sigma_{qq}(t)} \left(\frac{q+q'}{2} - \sigma_q(t) \right) (q-q') + \frac{i}{\hbar} \sigma_p(t) (q-q') \right], \end{aligned} \quad (14)$$

where

$$\sigma(t) \equiv \sigma_{qq}(t)\sigma_{pp}(t) - \sigma_{pq}^2(t) \quad (15)$$

is the determinant of the dispersion (correlation) matrix

$$M(t) = \begin{pmatrix} \sigma_{qq}(t) & \sigma_{pq}(t) \\ \sigma_{pq}(t) & \sigma_{pp}(t) \end{pmatrix} \quad (16)$$

and represents also the Schrödinger generalized uncertainty function [15].

For an initial Gaussian Wigner function (corresponding to a correlated coherent state (9)) the solution of Eq. (13) is

$$\begin{aligned} W(q, p, t) = & \frac{1}{2\pi\sqrt{\sigma(t)}} \exp \left\{ -\frac{1}{2\sigma(t)} [\sigma_{pp}(t)(q - \sigma_q(t))^2 + \right. \\ & \left. + \sigma_{qq}(t)(p - \sigma_p(t))^2 - 2\sigma_{pq}(t)(q - \sigma_q(t))(p - \sigma_p(t))] \right\}. \end{aligned} \quad (17)$$

In the case of a thermal bath we obtain the following steady state solution for $t \rightarrow \infty$ (we denote $\epsilon \equiv \frac{\hbar\omega}{2kT}$):

$$\begin{aligned} \langle q | \rho(\infty) | q' \rangle &= \left(\frac{m\omega}{\pi\hbar \coth \epsilon} \right)^{\frac{1}{2}} \times \\ &\times \exp \left\{ -\frac{m\omega}{4\hbar} \left[\frac{(q+q')^2}{\coth \epsilon} + (q-q')^2 \coth \epsilon \right] \right\}. \end{aligned} \quad (18)$$

In the long time limit we have also

$$W_\infty(q, p) = \frac{1}{\pi\hbar \coth \epsilon} \exp \left\{ -\frac{1}{\hbar \coth \epsilon} \left[m\omega q^2 + \frac{p^2}{m\omega} \right] \right\}. \quad (19)$$

Stationary solutions to the evolution equations obtained in the long time limit are possible as a result of a balance between the wave packet spreading induced by the Hamiltonian and the localizing effect of the Lindblad operators.

4. QUANTUM DECOHERENCE

As we stated in the Introduction, QD is a condition that has to be satisfied in order that a system could be considered as classical. This condition requires that the system should be in one of relatively permanent states (states that are least affected by the interaction of the system with the environment, called by Zurek “preferred states” in the environment induced superselection description [2, 3]) and the interference between different states should be negligible. This implies the destruction of off-diagonal elements representing coherences between quantum states in the density matrix, which is the QD phenomenon. This does not imply that the knowledge of the state of the system is necessarily precise, by contrary, in the general case we may have actually only a probabilistic description. For example, an initial pure state with a density matrix which contains nonzero off-diagonal terms can non-unitarily evolve into a final mixed state with a diagonal density matrix during the interaction with the environment, like in classical statistical mechanics. An isolated system has an unitary evolution and the coherence of the state is not lost: pure states evolve in time only to pure states. The loss of coherence can be achieved by introducing an interaction between the system and environment. The density matrix does not diagonalize exactly in position, but with a non-zero width, *i.e.* it is strongly peaked about $q = q'$ and very small for q far from q' .

Using new variables $\Sigma = (q + q')/2$ and $\Delta = q - q'$, the density matrix (14) can be rewritten as

$$\begin{aligned} \rho(\Sigma, \Delta, t) &= \sqrt{\frac{\alpha}{\pi}} \exp \left[-\alpha \Sigma^2 - \gamma \Delta^2 + i\beta \Sigma \Delta + \right. \\ &\left. + 2\alpha \sigma_q(t) \Sigma + i \left(\frac{\sigma_p(t)}{\hbar} - \beta \sigma_q(t) \right) \Delta - \alpha \sigma_q^2(t) \right], \end{aligned} \quad (20)$$

with the abbreviations

$$\alpha = \frac{1}{2\sigma_{qq}(t)}, \quad \gamma = \frac{\sigma(t)}{2\hbar^2\sigma_{qq}(t)}, \quad \beta = \frac{\sigma_{pq}(t)}{\hbar\sigma_{qq}(t)} \quad (21)$$

and the Wigner transform of the density matrix (20) is

$$W(q, p, t) = \frac{1}{2\pi\hbar} \sqrt{\frac{\alpha}{\gamma}} \times \exp\left\{-\frac{[\hbar\beta(q - \sigma_q(t)) - (p - \sigma_p(t))]^2}{4\hbar^2\gamma} - \alpha(q - \sigma_q(t))^2\right\}. \quad (22)$$

The representation-independent measure of the degree of QD [4] is given by the ratio of the dispersion $1/\sqrt{2\gamma}$ of the off-diagonal element $\rho(0, \Delta, t)$ to the dispersion $\sqrt{2/\alpha}$ of the diagonal element $\rho(\Sigma, 0, t)$

$$\delta_{QD} = \frac{1}{2} \sqrt{\frac{\alpha}{\gamma}}, \quad (23)$$

which in our case gives

$$\delta_{QD}(t) = \frac{\hbar}{2\sqrt{\sigma(t)}}. \quad (24)$$

The finite temperature Schrödinger generalized uncertainty function (15), calculated in Ref. [15], has the expression

$$\begin{aligned} \sigma(t) = & \frac{\hbar^2}{4} \left\{ e^{-4\lambda t} \left[1 - \left(\delta + \frac{1}{\delta(1-r^2)} \right) \coth \epsilon + \coth^2 \epsilon \right] + \right. \\ & + e^{-2\lambda t} \coth \epsilon \left[\left(\delta + \frac{1}{\delta(1-r^2)} - 2 \coth \epsilon \right) \frac{\omega^2 - \mu^2 \cos(2\Omega t)}{\Omega^2} + \right. \\ & \left. \left. + \left(\delta - \frac{1}{\delta(1-r^2)} \right) \frac{\mu \sin(2\Omega t)}{\Omega} + \frac{2r\mu\omega(1 - \cos(2\Omega t))}{\Omega^2 \sqrt{1-r^2}} \right] + \coth^2 \epsilon \right\}. \end{aligned} \quad (25)$$

In the limit of long times Eq. (25) yields

$$\sigma(\infty) = \frac{\hbar^2}{4} \coth^2 \epsilon, \quad (26)$$

so that we obtain

$$\delta_{QD}(\infty) = \frac{1}{\coth \frac{\hbar\omega}{2kT}}. \quad (27)$$

For high T we get

$$\delta_{QD}(\infty) = \frac{\hbar\Omega}{2kT}. \quad (28)$$

We see that δ_{QD} decreases, and therefore QD increases, with temperature, *i.e.*, the density matrix becomes more and more diagonal at higher T and the contributions of the off-diagonal elements get smaller and smaller. At the same time the degree of purity decreases and the degree of mixedness increases with T . For $T=0$ the asymptotic (final) state is pure and δ_{QD} reaches its initial maximum value 1. A pure state undergoing unitary evolution is highly coherent: it does not lose its coherence, *i.e.* off-diagonal coherences never vanish. $\delta_{QD} = 0$ when the quantum coherence is completely lost. So, when $\delta_{QD} = 1$ there is no QD and only if $\delta_{QD} < 1$ there is a significant degree of QD, when the magnitude of the elements of the density matrix in the position basis are peaked preferentially along the diagonal $q = q'$. When $\delta_{QD} \ll 1$ we have a strong QD.

For simplicity we consider zero values for the initial expectations values of coordinate and momentum and the expression (22) of the Wigner function becomes

$$W(q, p, t) = \frac{1}{2\pi\hbar} \sqrt{\frac{\alpha}{\gamma}} \exp\left[-\frac{(\hbar\beta q - p)^2}{4\hbar^2\gamma} - \alpha q^2\right]. \quad (29)$$

In coordinates $\hbar\beta q - p$ and q , $2\hbar\sqrt{\gamma}$ and $1/\sqrt{\alpha}$ are the lengths of the shorter and longer semi-axes of the 1σ contour and their product gives the area of the 1σ ellipse. We see from Eq. (23) that δ_{QD} is inversely proportional to this area. Besides this geometric interpretation, δ_{QD} is also connected with the linear entropy [19, 20].

We have seen that if the initial wave function is Gaussian, then the density matrix (14) and the Wigner function (17) remain Gaussian for all times (with time-dependent parameters which determine their amplitude and spread) and centered along the trajectory given by the solutions $\sigma_q(t)$ and $\sigma_p(t)$ of the dissipative equations of motion.

The degree of QD has an evolution which shows that in general QD increases with time and temperature. $\delta_{QD} < 1$ for long enough time, so that we can say that the considered system interacting with the thermal bath manifests QD. Dissipation promotes quantum coherences, whereas fluctuation (diffusion) reduces coherences and promotes QD. The balance of dissipation and fluctuation determines the final equilibrium value of δ_{QD} . The quantum system starts as a pure state, with a Wigner function well localized in phase space (Gaussian form).

This state evolves approximately following the classical trajectory (Liouville flow) in phase space and becomes a quantum mixed state during the irreversible process of QD.

5. SUMMARY AND CONCLUDING REMARKS

In the present paper we have studied QD with the Markovian equation of Lindblad in order to understand the quantum to classical transition for a system consisting of an one-dimensional harmonic oscillator in interaction with a thermal bath in the framework of the theory of open quantum systems based on quantum dynamical semigroups.

The role of QD became relevant in many interesting physical problems from field theory, atomic physics, quantum optics and quantum information processing, to which we can add material science, heavy ion collisions, quantum gravity and cosmology, condensed matter physics. Just to mention only a few of them: to understand the way in which QD favors the quantum to classical transition of density fluctuations; to study systems of trapped and cold atoms (or ions) which may offer the possibility of engineering the environment, like trapped atoms inside cavities, relation between decoherence and other cavity QED effects (such as Casimir effect); on mesoscopic scale, decoherence in the context of Bose-Einstein condensation [2, 3].

In many cases physicists are interested in understanding the specific causes of QD just because they want to prevent decoherence from damaging quantum states and to protect the information stored in quantum states from the degrading effect of the interaction with the environment. Thus, decoherence is responsible for washing out the quantum interference effects which are desirable to be seen as signals in some experiments. QD has a negative influence on many areas relying upon quantum coherence effects, such as quantum computation and quantum control of atomic and molecular processes. The physics of information and computation is such a case, where decoherence is an obvious major obstacle in the implementation of information-processing hardware that takes advantage of the superposition principle [21].

The study of classicality using QD leads to a deeper understanding of the quantum origins of the classical world. Much work has still to be done even to settle the interpretational questions, not to speak about answering them. Nevertheless, as a result of the progress made in the last two decades, the quantum to classical transition has become a subject of experimental investigations, while previously it was mostly a domain of philosophy [2, 3]. The issue of quantum to classical transition points to the necessity of a better understanding of open quantum systems. The Lindblad theory provides a selfconsistent treatment of damping as a general extension of quantum mechanics

to open systems and gives the possibility to extend the model of quantum Brownian motion. The obtained results in the framework of the Lindblad theory can be used for the description in more details of the connection between uncertainty, decoherence and correlations (entanglement) of open quantum systems with their environment.

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