QUASIPERIODIC PACKINGS OF ICOSAHEDRAL CLUSTERS OBTAINED BY PROJECTION

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The diffraction diagram of a quasicrystal admits as symmetry group a finite group G, and there exists a G-cluster $\mathcal{C}$ (a union of orbits of G) such that the quasicrystal can be regarded as a quasiperiodic packing of copies of $\mathcal{C}$, generally, partially occupied. On the other hand, by starting from a G-cluster $\mathcal{C}$ one can define in a canonical way an orthogonal representation of G in a higher dimensional space, decompose this space into the orthogonal sum of two G-invariant subspaces and use the strip projection method in order to define a pattern $\mathcal{Q}$ which can also be regarded as a quasiperiodic packing of copies of $\mathcal{C}$, generally, partially occupied. We show that any pattern defined in terms of the strip projection method can be re-defined as a multi-component model set by using, generally, a smaller dimensional superspace.

1. INTRODUCTION

The set of the Bragg peaks with intensity above a certain threshold occuring in the diffraction diagram of a quasicrystal is a discrete set, invariant under a finite group G, and the high-resolution electron microscopy suggests that the quasicrystal can be regarded as a packing of copies (generally, partially occupied) of a well-defined G-invariant atomic cluster $\mathcal{C}$.

From a mathematical point of view, the cluster $\mathcal{C}$ can be defined as a finite union of orbits of G, and there exists an algorithm [2, 3] which leads from $\mathcal{C}$ directly to a pattern $\mathcal{Q}$ which can be regarded as a union of interpenetrating partially occupied translations of $\mathcal{C}$ (the neighbours of each point $x \in \mathcal{Q}$ belong to the set $x+\mathcal{C}=\{x+y \mid y \in \mathcal{C}\}$). This algorithm, based on the strip projection method and group theory, represents an extended version of the model proposed by Katz & Duneau [9] and independently by Elser [8] for the icosahedral quasicrystals.

The dimension of the superspace used in the case of a two or three-shell icosahedral cluster is rather large, and the main purpose of this paper is to present a way to reduce this dimension. It is based on the notion of *multi-component model set*, an extension of the notion of *model set*, proposed by Baake and Moody [1]. Some examples are presented in order to illustrate the theoretic considerations.

2. STRIP PROJECTION METHOD AND MULTI-COMPONENT MODEL SETS

Let $\mathbb{E}_k = (\mathbb{R}^k, \langle,\rangle)$ be the usual *k*-dimensional Euclidean space, E and $E^\perp$ be two orthogonal subspaces such that $\mathbb{E}_k = E \oplus E^\perp$, and let

$$L = \kappa\mathbb{Z}^k \qquad \mathbb{K} = [0, \kappa]^k = \{(x_1, x_2, ..., x_k) \,|\, 0 \le x_i \le \kappa \text{ for all } i\} \tag{1}$$

where $\kappa \in (0, \infty)$ is a fixed constant. For each $x \in \mathbb{E}_k$ there exist $x^{\|} \in E$ and $x^\perp \in E^\perp$ uniquely determined such that $x = x^{\|} + x^\perp$. The mappings

$$\pi : \mathbb{E}_k \longrightarrow \mathbb{E}_k : x \mapsto \pi x = x^{\|} \qquad \pi^\perp : \mathbb{E}_k \longrightarrow \mathbb{E}_k : x \mapsto \pi^\perp x = x^\perp \tag{2}$$

are the corresponding orthogonal projectors.

By using the bounded set $K = \pi^\perp(\mathbb{K})$ we define in terms of the *strip projection method* [9] the discrete set

$$\mathcal{Q} = \left\{\pi x \,\middle|\, x \in \mathbb{L},\ \pi^\perp x \in K\right\} \tag{3}$$

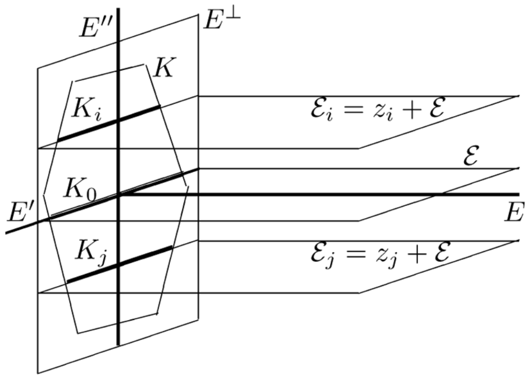
formed by the projection on E of all the points of $\mathbb{L}$ lying in the *strip* $K + E = \{x + y \,|\, x \in K, y \in E\}$.

It is known [11] that any $\mathbb{Z}$-module $M \subset \mathbb{R}^l$ is the direct sum of a lattice M_d of rank d and a $\mathbb{Z}$-module M_s dense in a vector subspace of dimension s, where $d + s$ is the dimension of the subspace generated by L in $\mathbb{R}^l$. In view of this result the $\mathbb{Z}$-module $\pi^\perp(\mathbb{L})$ is the direct sum $\pi^\perp(\mathbb{L}) = \mathcal{L}' \oplus \mathcal{D}$ of a lattice $\mathcal{D}$ of rank d and a $\mathbb{Z}$-module $\mathcal{L}'$ dense in a subspace $E' \subset E^\perp$ of dimension s, where $d + s = \dim E^\perp$. In this decomposition the space E' is uniquely determined and we denote by E'' its orthogonal complement in $E^\perp$

$$E'' = \left\{x \in E^\perp \,\middle|\, \langle x, y\rangle = 0 \text{ for any } y \in E'\right\}.$$

We get $\mathbb{E}_k = E \oplus E' \oplus E''$ (see Fig. 1). For each $x \in \mathbb{E}_k$ there exist $x^{\|} \in E$, $x' \in E'$ and $x'' \in E''$ uniquely determined such that $x = x^{\|} + x' + x''$. The mappings $\pi' : \mathbb{E}_k \longrightarrow \mathbb{E}_k$, $\pi' x = x'$ and $\pi'' : \mathbb{E}_k \longrightarrow \mathbb{E}_k$, $\pi'' x = x''$ are the orthogonal projectors corresponding to E' and E''.

Fig. 1. – The decompositions $\mathbb{E}_k = = E \oplus E^{\perp} = E \oplus E' \oplus E'' = \mathcal{E} \oplus E''$.



One can prove [4] that the projection $\mathcal{L} = (\pi + \pi')(\mathbb{L})$ of the lattice $\mathbb{L}$ on the space $\mathcal{E} = E \oplus E'$ is a lattice in $\mathcal{E}$, π restricted to $\mathcal{L}$ is injective and $\pi'(\mathcal{L})$ is dense in E'. It follows that the collection of spaces and mappings

$$\begin{array}{ccccc} \pi x \leftarrow x : E & \xleftarrow{\pi} & \mathcal{E} & \xrightarrow{\pi'} & E' : x \to \pi' x \\ & & \cup & & \\ & & \mathcal{L} & & \end{array} \tag{4}$$

is a cut and project scheme [1, 10].

The lattice $L = \mathbb{L} \cap \mathcal{E}$ is a sublattice of $\mathcal{L}$, and necessarily $[\mathcal{L} : L]$ is finite. The projection $\mathbb{L}'' = \pi''(\mathbb{L})$ of $\mathbb{L}$ on E'' is a discrete countable set. Let $\mathcal{Z} = \{z_i \mid i \in \mathbb{Z}\}$ be a subset of $\mathbb{L}$ such that $\mathbb{L}'' = \pi''(\mathcal{Z})$ and $\pi'' z_i \neq \pi'' z_j$ for $i \neq j$. The lattice $\mathbb{L}$ is contained in the union $\bigcup_{i \in \mathbb{Z}} \mathcal{E}_i$ of the cosets $\mathcal{E}_i = z_i + \mathcal{E} = = \{z_i + x \mid x \in \mathcal{E}\}$. Since $\mathbb{L} \cap \mathcal{E}_i = z_i + L$ the set $\mathcal{L}_i = (\pi + \pi')(\mathbb{L} \cap \mathcal{E}_i) = = (\pi + \pi') z_i + L$ is a coset of L in $\mathcal{L}$ for any $i \in \mathbb{Z}$.

Only for a finite number of cosets $\mathcal{E}_i$ the intersection

$$K_i = K \cap \mathcal{E}_i = \pi^{\perp}(K \cap \mathcal{E}_i) \subset \pi'' z_i + E' \tag{5}$$

is non-empty. By changing the indexation of the elements of $\mathcal{Z}$ if necessary, we can assume that the subset $\mathcal{K}_i = \pi'(K_i) = \pi'(\mathbb{K} \cap \mathcal{E}_i)$ of E' has a non-empty interior only for $i \in \{1, \ldots, m\}$. The 'polyhedral' set $\mathcal{K}_i$ satisfies the conditions:

(a) $\mathcal{K}_i \subset E'$ is compact;

(b) $\mathcal{K}_i = \overline{\mathrm{int}(\mathcal{K}_i)}$;

(c) The boundary of $\mathcal{K}_i$ has Lebesgue measure 0 for any $i \in \{1, \ldots, m\}$. This allows us to define the multi-component model set [1]

$$\Lambda = \bigcup_{i=1}^{m} \left\{ \pi x \,\middle|\, x \in \mathcal{L}_i,\ \pi' x \in \mathcal{K}_i \right\}. \tag{6}$$

One can remark that $\Lambda = \mathcal{Q}$, and hence we have re-defined our pattern $\mathcal{Q}$ as a multi-component model set by using the superspace $\mathcal{E}$ of dimension, generally, smaller than the dimension k of the initial superspace $\mathbb{E}_k$. The main difficulty in this new approach is the determination of the 'atomic surfaces' $\mathcal{K}_i$.

3. PACKINGS OF *G*-CLUSTERS OBTAINED BY PROJECTION

In this section we review our method to obtain packings of clusters by projection. It is a direct generalization of the model proposed by Katz & Duneau [9] and independently by Elser [8] for icosahedral quasicrystals.

Let $\{g : \mathbb{E}_n \longrightarrow \mathbb{E}_n \mid g \in G\}$ be an orthogonal $\mathbb{R}$-irreducible faithful representation of a finite group *G* in the 'physical' space $\mathbb{E}_n$ and let $S \subset \mathbb{E}_n$ be a finite non-empty set which does not contain the null vector. Any finite union of orbits of *G* is called a *G-cluster*. Particularly,

$$\mathcal{C} = \bigcup_{r \in S} Gr \cup \bigcup_{r \in S} G(-r) = \{e_1, e_2, \ldots, e_k, -e_1, -e_2, \ldots, -e_k\} \tag{7}$$

where $Gr = \{gr \mid g \in G\}$, is the *G*-cluster symmetric with respect to the origin generated by *S*.

Let $e_i = (e_{i1}, e_{i2}, \ldots, e_{in})$, and let $\varepsilon_1 = (1, 0, \ldots, 0)$, $\varepsilon_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $\varepsilon_k = (0, \ldots, 0, 1)$ be the canonical basis of $\mathbb{E}_k$. For each $g \in G$, there exist the numbers s_1^g, s_2^g, $\ldots$, $s_k^g \in \{-1; 1\}$ and a permutation of the set $\{1, 2, \ldots, k\}$ denoted also by g such that,

$$g e_j = s_{g(j)}^g e_{g(j)} \qquad \text{for all } j \in \{1, 2, \ldots, k\}. \tag{8}$$

Theorem 1. [2,3] *The group G can be identified with the group of permutations*

$$\left\{ \mathcal{C} \longrightarrow \mathcal{C} : r \mapsto gr \,\middle|\, g \in G \right\}$$

and the formula

$$g \varepsilon_j = s_{g(j)}^g \varepsilon_{g(j)} \qquad \textit{for all } j \in \{1, 2, \ldots, k\} \tag{9}$$

defines the orthogonal representation

$$g(x_1, x_2, \ldots, x_k) = (s_1^g x_{g^{-1}(1)}, s_2^g x_{g^{-1}(2)}, \ldots, s_k^g x_{g^{-1}(k)}) \tag{10}$$

of G in $\mathbb{E}_k$.

Theorem 2. [2, 3] *The subspaces*

$$\begin{aligned} E &= \{(<r, e_1>, <r, e_2>, \ldots, <r, e_k>) \mid r \in \mathbb{E}_n\} \\ E^{\perp} &= \left\{(x_1, x_2, \ldots, x_k) \in \mathbb{E}_k \,\middle|\, \sum_{i=1}^{k} x_i e_i = 0\right\} \end{aligned} \tag{11}$$

of $\mathbb{E}_k$ *are G-invariant, orthogonal, and* $\mathbb{E}_k = E \oplus E^{\perp}$.

Theorem 3. [2, 3] *The vectors* $v_1 = \varrho(e_{11}, e_{21}, \ldots, e_{k1})$, ..., $v_n = \varrho(e_{1n}, e_{2n}, \ldots, e_{kn})$, *where* $\varrho = 1/\sqrt{(e_{11})^2 + (e_{21})^2 + \ldots + (e_{k1})^2}$ *form an orthonormal basis of E.*

Theorem 4. [2, 3] *The subduced representation of G in E is equivalent with the representation of G in* $\mathbb{E}_n$, *and the isomorphism*

$$\varphi : \mathbb{E}_n \longrightarrow E \qquad \varphi(r) = (\varrho <r, e_1>, \varrho <r, e_2>, \ldots, \varrho <r, e_k>) \tag{12}$$

with the property $\varphi(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ *allows us to identify the 'physical' space* $\mathbb{E}_n$ *with the subspace E of* $\mathbb{E}_k$.

Theorem 5. [2, 3] *The matrix of the orthogonal projector* $\pi : \mathbb{E}_k \longrightarrow \mathbb{E}_k$ *corresponding to E in the basis* $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k\}$ *is*

$$\pi = \varrho^2 \begin{pmatrix} \langle e_1, e_1\rangle & \langle e_1, e_2\rangle & \ldots & \langle e_1, e_k\rangle \\ \langle e_2, e_1\rangle & \langle e_2, e_2\rangle & \ldots & \langle e_2, e_k\rangle \\ \ldots & \ldots & \ldots & \ldots \\ \langle e_k, e_1\rangle & \langle e_k, e_2\rangle & \ldots & \langle e_k, e_k\rangle \end{pmatrix}. \tag{13}$$

Let $\kappa = 1/\varrho$, $\mathbb{L} = \kappa\mathbb{Z}^k$, $\mathbb{K} = [0, \kappa]^k$, and let $K = \pi^{\perp}(\mathbb{K})$, where $\pi^{\perp} : \mathbb{E}_k \longrightarrow \mathbb{E}_k$, $\pi^{\perp}x = x - \pi x$ is the orthogonal projector corresponding to $E^{\perp}$.

Theorem 6. [2, 3] *The* $\mathbb{Z}$-*module* $\mathbb{L} \subset \mathbb{E}_k$ *is G-invariant,* $\pi(\kappa\varepsilon_i) = \varphi(e_i)$, *that is,* $\pi(\kappa\varepsilon_i) = e_i$ *if we take into consideration the identification* $\varphi : \mathbb{E}_n \longrightarrow E$, *and*

$$\pi(\mathbb{L}) = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \ldots + \mathbb{Z}e_k. \tag{14}$$

The pattern defined by using the strip projection method [9]

$$\mathcal{Q}=\left\{\pi x \,\middle|\, x\in\mathbb{L},\ \pi^{\perp}x\in K\right\} \tag{15}$$

can be regarded as a union of interpenetrating partially occupied copies of $\mathcal{C}$. For each point $\pi x\in\mathcal{Q}$ the set of all the arithmetic neighbours of πx

$$\{\pi y \mid y\in\{x+\kappa\varepsilon_1,\ldots,x+\kappa\varepsilon_k,x-\kappa\varepsilon_1,\ldots,x-\kappa\varepsilon_k\},\pi^{\perp}y\in K\}$$

is contained in the translated copy

$$\{\pi x+e_1,\ldots,\pi x+e_k,\pi x-e_1,\ldots,\pi x-e_k\}=\pi x+\mathcal{C}$$

of the G-cluster $\mathcal{C}$. The fully occupied clusters occuring in $\mathcal{Q}$ correspond to the points $x\in\mathbb{L}$ satisfying the condition [9]

$$\pi^{\perp}x\in K\cap\bigcap_{i=1}^{k}(\pi^{\perp}(\kappa\varepsilon_i)+K)\cap\bigcap_{i=1}^{k}(-\pi^{\perp}(\kappa\varepsilon_i)+K). \tag{16}$$

Generally, only a small part of the clusters occuring in $\mathcal{Q}$ can be fully occupied. A fragment of $\mathcal{Q}$ can be obtained by using, for example, the algorithm presented in [12].

The method presented in the previous section allows to re-define $\mathcal{Q}$ as a multi-component model set by using, generally, a smaller dimensional superspace. Some examples are presented in the next sections.

4. A QUASIPERIODIC PACKING OF DECAGONS

The relations

$$a(x,y)=(cx-sy,\,sx+cy) \qquad b(x,y)=(x,-y) \tag{17}$$

where $c=\cos(\pi/5)=(1+\sqrt{5})/4$, $s=\sin(\pi/5)=\sqrt{10-2\sqrt{5}}/4$ define the usual two-dimensional representation of the dihedral group

$$D_{10}=\left\langle a,b \,\middle|\, a^{10}=b^2=(ab)^2=e\right\rangle.$$

Let $\varepsilon_1=(1,0,\ldots,0)$, $\varepsilon_2=(0,1,0,...,0)$, $\ldots$, $\varepsilon_5=(0,\ldots,0,1)$ be the canonical basis of $\mathbb{E}_5$, and let $c'=\cos(2\pi/5)=(\sqrt{5}-1)/4$, $s'=\sin(2\pi/5)=\sqrt{10+2\sqrt{5}}/4$. The D_{10}-cluster (containing only one orbit) generated by the set $S=\{(1,0)\}$ is

$$\mathcal{C}=D_{10}(1,0)=\{e_1,e_2,e_3,e_4,e_5,-e_1,-e_2,-e_3,-e_4,-e_5\}$$

where $e_1=(1,0)$, $e_2=(c',s')$, $e_3=(-c,s)$, $e_4=(-c,-s)$, $e_5=(c',-s')$. It is formed by the vertices of a regular decagon.

The action of a and b on $\mathcal{C}$ is described by the signed permutations

$$a=\begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ -e_4 & -e_5 & -e_1 & -e_2 & -e_3 \end{pmatrix} \qquad b=\begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ e_1 & e_5 & e_4 & e_3 & e_2 \end{pmatrix} \tag{18}$$

and the corresponding transformations $a, b : \mathbb{E}_5 \longrightarrow \mathbb{E}_5$

$$a=\begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 \\ -\varepsilon_4 & -\varepsilon_5 & -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 \end{pmatrix} \qquad b=\begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 \\ \varepsilon_1 & \varepsilon_5 & \varepsilon_4 & \varepsilon_3 & \varepsilon_2 \end{pmatrix} \tag{19}$$

generate the orthogonal representation of D_{10} in $\mathbb{E}_5$

$$\begin{aligned} a(x_1, x_2, x_3, x_4, x_5) &= (-x_3, -x_4, -x_5, -x_1, -x_2) \\ b(x_1, x_2, x_3, x_4, x_5) &= (x_1, x_5, x_4, x_3, x_2). \end{aligned} \tag{20}$$

The vectors $v_1 = \varrho(1, c', -c, -c, c')$, $v_2 = \varrho(0, s', s, -s, -s')$, where $\varrho = \sqrt{2/5}$, form an orthonormal basis of the D_{10}-invariant subspace

$$E = \{(<r, e_1>, <r, e_2>, \ldots, <r, e_5>) \mid r \in \mathbb{E}_2\} \tag{21}$$

and the isometry (which is an isomorphism of representations)

$$\varphi : E_2 \longrightarrow E : r \mapsto (\varrho <r, e_1>, \varrho <r, e_2>, \ldots, \varrho <r, e_5>) \tag{22}$$

with the property $\varphi(\alpha, \beta) = \alpha v_1 + \beta v_2$ allows us to identify the physical space $\mathbb{E}_2$ with the subspace E of $\mathbb{E}_5$. The matrices of the orthogonal projectors π, $\pi^\perp : \mathbb{E}_5 \longrightarrow \mathbb{E}_5$ corresponding to E and

$$E^\perp = \{x \in \mathbb{E}_5 \mid \langle x, y \rangle = 0 \quad \text{for all } \; y \in E\} \tag{23}$$

in the basis $\{\varepsilon_1, ..., \varepsilon_5\}$ are

$$\pi = \mathcal{M}(2/5, -\tau'/5, -\tau/5) \qquad \pi^\perp = \mathcal{M}(3/5, \tau'/5, \tau/5) \tag{24}$$

where $\tau = (1+\sqrt{5})/2$, $\tau' = (1-\sqrt{5})/2$ and

$$\mathcal{M}(\alpha, \beta, \gamma) = \begin{pmatrix} \alpha & \beta & \gamma & \gamma & \beta \\ \beta & \alpha & \beta & \gamma & \gamma \\ \gamma & \beta & \alpha & \beta & \gamma \\ \gamma & \gamma & \beta & \alpha & \beta \\ \beta & \gamma & \gamma & \beta & \alpha \end{pmatrix}. \tag{25}$$

Let $\kappa = 1/\varrho = \sqrt{5/2}$, $\mathbb{L} = \kappa\mathbb{Z}^5$, $\mathbb{K} = [0, \kappa]^5$, and let $K = \pi^\perp(\mathbb{K})$. The pattern defined in terms of the strip projection method

$$\mathcal{Q} = \left\{ \pi x \middle| x \in \mathbb{L},\ \pi^{\perp} x \in K \right\} \tag{26}$$

is the set of vertices of a *Penrose (singular) tiling* [9]. For each $i \in \{1, 2, 3, 4, 5\}$ we have $\pi(\kappa\varepsilon_i) = \varphi(e_i)$, that is, $\pi(\kappa\varepsilon_i) = e_i$ if we use the identification of $\mathbb{E}_2$ with E. Since the arithmetic neighbours of each point $\pi x \in \mathcal{Q}$ belong to the set $\pi x + \mathcal{C}$ we can regard $\mathcal{Q}$ as a quasiperiodic packing of partially occupied copies of the D_{10}-cluster $\mathcal{C}$, that is, a quasiperiodic packing of decagons.

We can re-define $\mathcal{Q}$ as a multi-component model set by using the general method presented in section 2. In this case, we have to use the decomposition $E^{\perp} = E' \oplus E''$ corresponding to the orthogonal projectors

$$\pi' = \mathcal{M}(2/5, -\tau/5, -\tau'/5) \qquad \pi'' = \mathcal{M}(1/5, 1/5, 1/5).$$

We get

$$\mathcal{E} = E \oplus E' = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{E}_5 \mid x_1 + x_2 + x_3 + x_4 + x_5 = 0\} \tag{27}$$

$$E'' = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{E}_5 \mid x_1 = x_2 = x_3 = x_4 = x_5\} \tag{28}$$

$$\mathcal{L} = (\pi + \pi')(\mathbb{L}) = \mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}w_3 + \mathbb{Z}w_4 \tag{29}$$

$$L = \mathbb{L} \cap \mathcal{E} = 5\mathcal{L} = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{L} \mid x_1 + x_2 + x_3 + x_4 + x_5 = 0\} \tag{30}$$

where

$$\begin{aligned} w_1 &= \frac{1}{\sqrt{10}}(4, -1, -1, -1, -1) \qquad & w_2 &= \frac{1}{\sqrt{10}}(-1, 4, -1, -1, -1) \\ w_3 &= \frac{1}{\sqrt{10}}(-1, -1, 4, -1, -1) \qquad & w_4 &= \frac{1}{\sqrt{10}}(-1, -1, -1, 4, -1). \end{aligned} \tag{31}$$

We can choose $z_j = (\kappa j, 0, 0, 0, 0)$ since $\pi'' z_i \neq \pi'' z_j$ for $i \neq j$,

$$\mathcal{E}_j = z_j + \mathcal{E} = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{E}_5 \mid x_1 + x_2 + x_3 + x_4 + x_5 = \kappa j\}$$

and

$$\mathbb{L} \subset \bigcup_{j \in \mathbb{Z}} \mathcal{E}_j.$$

The set $\mathbb{K} \cap \mathcal{E}_i$ is non-empty only for $i \in \{0, 1, 2, 3, 4, 5\}$, but $\mathcal{K}_i = \pi'(\mathbb{K} \cap \mathcal{E}_i)$ has non-empty interior only for $i \in \{0, 1, 2, 3, 4\}$. The set K_1 is the regular pentagon with the vertices

$$\pi'(\kappa, 0, 0, 0, 0) = \frac{1}{\sqrt{10}}(2, -\tau, -\tau', -\tau', -\tau)$$

$$\begin{aligned}
\pi'(0,\kappa,0,0,0) &= \frac{1}{\sqrt{10}}(-\tau, 2, -\tau, -\tau', -\tau') \\
\pi'(0,0,\kappa,0,0) &= \frac{1}{\sqrt{10}}(-\tau', -\tau, 2, -\tau, -\tau') \\
\pi'(0,0,0,\kappa,0) &= \frac{1}{\sqrt{10}}(-\tau', -\tau', -\tau, 2, -\tau) \\
\pi'(0,0,0,0,\kappa) &= \frac{1}{\sqrt{10}}(-\tau, -\tau', -\tau', -\tau, 2)
\end{aligned} \tag{32}$$

$\mathcal{K}_2 = -\tau\mathcal{K}_1$, $\mathcal{K}_3 = \tau\mathcal{K}_1$, $\mathcal{K}_4 = -\mathcal{K}_1$, and we can re-define the pattern $\mathcal{Q}$ as the multi-component model set

$$\mathcal{Q} = \bigcup_{i=1}^{4} \left\{ \pi x \,\middle|\, x \in \mathcal{L}_i,\ \pi' x \in \mathcal{K}_i \right\} \tag{33}$$

where $\mathcal{L}_j = (\pi + \pi')z_j + L = jw_1 + L$. This definition is directly related to de Bruijn's definition [10].

5. A QUASIPERIODIC PACKING OF ICOSAHEDRA

The icosahedral group $Y = 235 = \langle a,\ b \mid a^5 = b^2 = (ab)^3 = I \rangle$ has five irreducible non-equivalent representations. The rotations $a, b : \mathbb{E}_3 \longrightarrow \mathbb{E}_3$

$$\begin{aligned}
a(\alpha,\beta,\gamma) &= \left(\frac{\tau-1}{2}\alpha - \frac{\tau}{2}\beta + \frac{1}{2}\gamma, \frac{\tau}{2}\alpha + \frac{1}{2}\beta + \frac{\tau-1}{2}\gamma, -\frac{1}{2}\alpha + \frac{\tau-1}{2}\beta + \frac{\tau}{2}\gamma \right) \\
b(\alpha,\beta,\gamma) &= (-\alpha, -\beta, \gamma)
\end{aligned} \tag{34}$$

defines an irreducible three-dimensional representation of Y

Let $\varepsilon_1 = (1, 0, \ldots, 0)$, $\varepsilon_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $\varepsilon_6 = (0, \ldots, 0, 1)$ be the canonical basis of $\mathbb{E}_6$. The points of the one-shell Y-cluster

$$\mathcal{C} = Y(1, \tau, 0) = \{e_1, e_2, \ldots, e_6, -e_1, -e_2, \ldots, -e_6\}$$

where

$$\begin{aligned}
&e_1 = (1, \tau, 0) \quad && e_3 = (-\tau, 0, 1) \quad && e_5 = (\tau, 0, 1) \\
&e_2 = (-1, \tau, 0) \quad && e_4 = (0, -1, \tau) \quad && e_6 = (0, 1, \tau)
\end{aligned} \tag{35}$$

are the vertices of a regular icosahedron. The action of a and b on the set $\mathcal{C}$ is described by the signed permutations

$$a=\begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ e_2 & e_3 & e_4 & e_5 & e_1 & e_6 \end{pmatrix} \qquad b=\begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ -e_1 & -e_2 & e_5 & e_6 & e_3 & e_4 \end{pmatrix} \tag{36}$$

and the corresponding transformations $a, b : \mathbb{E}_6 \longrightarrow \mathbb{E}_6$

$$a=\begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_6 \\ \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_1 & \varepsilon_6 \end{pmatrix} \qquad b=\begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_6 \\ -\varepsilon_1 & -\varepsilon_2 & \varepsilon_5 & \varepsilon_6 & \varepsilon_3 & \varepsilon_4 \end{pmatrix} \tag{37}$$

generate the orthogonal representation of Y in $\mathbb{E}_6$

$$\begin{aligned} a(x_1, x_2, x_3, x_4, x_5, x_6) &= (x_5, x_1, x_2, x_3, x_4, x_6) \\ b(x_1, x_2, x_3, x_4, x_5, x_6) &= (-x_1, -x_2, x_5, x_6, x_3, x_4). \end{aligned} \tag{38}$$

The vectors

$$v_1 = \varrho(1, -1, -\tau, 0, \tau, 0) \qquad v_2 = \varrho(\tau, \tau, 0, -1, 0, 1) \qquad v_3 = \varrho(0, 0, 1, \tau, 1, \tau) \tag{39}$$

where $\varrho = 1/\sqrt{4+2\tau}$, form an orthonormal basis of the Y-invariant subspace

$$E = \left\{ \left(\langle r, e_1 \rangle, \langle r, e_2 \rangle, \dots, \langle r, e_6 \rangle \right) \mid r \in \mathbb{E}_3 \right\}. \tag{40}$$

The isometry

$$\varphi : E_3 \longrightarrow E : r \mapsto \left(\varrho \langle r, e_1 \rangle, \varrho \langle r, e_2 \rangle, \dots, \varrho \langle r, e_6 \rangle \right) \tag{41}$$

which is an isomorphism of representations [2, 3] of Y with the property $\varphi(\alpha, \beta, \gamma) = \alpha v_1 + \beta v_2 + \gamma v_3$ allows us to identify the ‘physical’ space $\mathbb{E}_3$ with E.

The matrices of the orthogonal projectors π, $\pi^\perp : \mathbb{E}_6 \longrightarrow \mathbb{E}_6$ corresponding to E and $E^\perp = \left\{ x \in \mathbb{E}_6 \mid \langle x, y \rangle = 0 \text{ for all } y \in E \right\}$ in the basis $\{\varepsilon_1, \dots, \varepsilon_6\}$ are

$$\pi = \mathcal{M}(1/2, \sqrt{5}/10) \qquad \pi^\perp = \mathcal{M}(1/2, -\sqrt{5}/10) \tag{42}$$

where

$$\mathcal{M}(\alpha, \beta) = \begin{pmatrix} \alpha & \beta & -\beta & -\beta & \beta & \beta \\ \beta & \alpha & \beta & -\beta & -\beta & \beta \\ -\beta & \beta & \alpha & \beta & -\beta & \beta \\ -\beta & -\beta & \beta & \alpha & \beta & \beta \\ \beta & -\beta & -\beta & \beta & \alpha & \beta \\ \beta & \beta & \beta & \beta & \beta & \alpha \end{pmatrix}. \tag{43}$$

Let $\kappa = 1/\varrho$, $\mathbb{L} = \kappa\mathbb{Z}^6$, $\mathbb{K} = [0, \kappa]^6$, and let $K = \pi^\perp(\mathbb{K})$. The pattern defined in terms of the strip projection method

$$\mathcal{Q} = \left\{ \pi x \mid x \in \mathcal{L},\ \pi^\perp x \in K \right\} \tag{44}$$

is the set of vertices of an *Ammann (singular) tiling* [9]. It can be regarded as a quasiperiodic packing of interpenetrating icosahedra. Only a very small part of the icosahedra occurring in this pattern are fully occupied [9].

6. QUASIPERIODIC PACKINGS OF ICOSAHEDRAL CLUSTERS

If in the previous example we replace the starting cluster $Y(1, \tau, 0)$ by the two-shell Y-cluster

$$\mathcal{C} = Y\{\alpha(1, \tau, 0),\ \beta(1, 1, 1)\} = Y(\alpha, \alpha\tau, 0) \cup Y(\beta, \beta, \beta) = \{e_1, \ldots, e_{16}, -e_1, \ldots, -e_{16}\}$$

where α, β are rational positive numbers, then we get a pattern $\mathcal{Q}$ such that the arithmetic neighbours of each point $\pi x \in \mathcal{Q}$ are distributed [3] on two shells, namely, on the vertices of a regular icosahedron of radius $\alpha\sqrt{\tau + 2}$ and on the vertices of a regular dodecahedron of radius $\beta\sqrt{3}$. The pattern $\mathcal{Q}$ can be regarded as a quasiperiodic packing of interpenetrating copies of $\mathcal{C}$. Only a small part of the copies of the cluster $\mathcal{C}$ occurring in $\mathcal{Q}$ can be fully occupied. We think that the frequency of occurrence of the fully occupied icosahedra may be much greater in this pattern than in the Ammann pattern (44). The pattern $\mathcal{Q}$ can be re-defined [6] as a multi-component model set by using a 6-dimensional superspace, but we have to determine some rather complicated 'atomic surfaces' $\mathcal{K}_i$.

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