

# INTEGRABILITY AND ALGEBRAIC ENTROPY FOR DISCRETE DYNAMICAL SYSTEMS

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We review the results of the paper [1] about the degree growth of the iterates of the initial conditions for a class of third-order integrable mappings which result from the coupling of a discrete Painlevé equation to an homographic mapping. We show that the degree grows like  $n^3$ . In the special cases where the mapping satisfies the singularity confinement requirement we find a slower, quadratic growth. Finally we present a method for the construction of integrable  $N$ th-order mappings with degree growth  $n^N$ .

## 1. INTRODUCTION

The integrability of a discrete system is related to the slow growth of some characteristic property. For rational mappings what turned out to be an easy-to-use and reliable integrability criterion was the growth of the degree of the iterates of some initial condition. A quantity that can be the most easily computed is the degree of the numerators or denominators of (the irreducible forms of) the iterates. This computation can be performed either by introducing homogeneous coordinates and computing the homogeneity degree or in the usual coordinates by obtaining the maximum of the degrees of the numerator and denominator. The seminal ideas for this approach are due to Arnold [2] and Veselov [3]. They were made quantitative by Viallet and collaborators [4, 5], leading to the introduction of the notion of algebraic entropy. The latter is defined as

$$E = \lim_{n \rightarrow \infty} \log(d_n)/n$$

where  $d_n$  is the degree of the  $n$ -th iterate. A generic, nonintegrable, mapping leads to exponential growth of the degrees of the iterates and thus has a nonzero algebraic entropy, while an integrable mapping has zero algebraic entropy. The reason why the degree growth of an integrable mapping is not maximal lies in the fact that, during the successive iterations, the same polynomial factors appear in the numerator and the denominator of the fraction that represents the  $n$ th iterate of some initial condition and thus cancel out. This factor cancellation is at

the origin of the property which characterises a large class of integrable mappings and which we have dubbed singularity confinement [6]: if during the iterations the dependent variable takes a value which corresponds to a root of a polynomial factor this may lead to a singularity, which, however, will eventually disappear when the appropriate polynomial factor is cancelled out.

In a series of works [7, 8, 9], the study of the degree growth for mappings of the *second* order has been refined. Here are the main conclusions:

- Mappings which are integrable through spectral techniques (as well as the autonomous limits of these mappings) have a degree growth of the  $n$ th iterate which goes like  $n^2$ . (Incidentally, all these mappings do satisfy the singularity confinement criterion which led to the formulation of the conjecture that this criterion may be sufficient for integrability through spectral techniques). From a more rigorous point of view some results do exist, confirming the quadratic growth. In [10] it was shown by Bellon that second-order autonomous mappings which possess a constant of motion have necessarily a quadratic degree growth. More recently, Takenawa [11] has shown that all the discrete Painlevé equations which can be embedded in the Weyl group degeneration scheme, proposed by Sakai [12] and Grammaticos *et al.* [13], have also quadratic degree growth.
- Mappings which are linearisable have degree growth slower than quadratic. Projective mappings have no growth at all (because of the underlying linear system). Mappings which are linearisable in a more complicated way (derivative of discrete Riccati, Gambier-type coupling of homographic mappings in cascade or the new linearisation method introduced in [14]) have linear growth. While for simple cases, like the discrete Riccati derivative, one can compute explicitly the degree growth (and confirm the result  $d_n \propto n$ ) the situation is more complicated for the other linearisable systems. Still using the algebraic geometrical approach introduced in [15], it is possible to compute explicitly [16] the growth of linearisable systems confirming the previously obtained results.

At this point a further remark is in order. In the results mentioned above we do not distinguish between autonomous and nonautonomous mappings. As a matter of fact, it turned out that, given an autonomous integrable mapping with a given growth, its *integrable* nonautonomous extension corresponds to the *same* growth. This remark makes possible the introduction of a simple strategy for the investigation of discrete integrability using the degree-growth techniques: start from an autonomous mapping and implement all the constraints which lead to nonexponential growth and then deautonomise by requiring that the degrees of the iterates be exactly the same as in the autonomous case.

## 2. EXAMPLE OF AN INTEGRABLE HIGHER ORDER MAPPING

While the domain of second-order mappings has been rather exhaustively explored, very little is known on higher-order systems. In [17] it was shown that the  $N$ th order Gambier mapping leads to a degree growth of  $n^{N-1}$ . Based on this result we conjectured that the generic *integrable*  $N$ th order mapping should have a growth of  $n^N$ . However, this was not materialised by explicit computations of degree growth. The main difficulty is the scarcity of integrable higher-order mappings. In this paper we intend to examine the case of third-order mappings where some results do exist and compare the computational results to the  $n^3$  growth predicted by the conjecture above.

The first mapping we are going to treat is the  $q$ -deformed discrete Painlevé I proposed by Nijhoff [18]:

$$\beta q^{-n}(x_{n+1} - q^2 x_{n-2}) = q \frac{(n)_q + \alpha q^n}{x_n} - \frac{(n-1)_q + \alpha q^{n-1}}{x_{n-1}} \quad (1)$$

where  $(n)_q = (q^n - 1)/(q - 1)$ . This mapping is integrable and Nijhoff has explicitly constructed its Lax pair. In order to study the degree growth of this mapping we introduce the following initial conditions:  $x_0 = r$ ,  $x_1 = s$ ,  $x_2 = p/q$  and assign a degree 0 to  $r$ ,  $s$  and a degree 1 to  $p$ ,  $q$ . (Other choices do exist, of course, and the details of the degrees do depend on the one chosen. However the growth itself is independent of these details). We obtain the following sequence of homogeneity degrees in  $p$ ,  $q$  of the numerator/denominator of  $x_n$ :

$$d_n = 0, 0, 1, 2, 4, 5, 7, 10, 12, 15, 19, 22, 26, 31, 35, 40, \dots$$

The growth is manifestly quadratic. This can be easily assessed if one computes the successive differences between the degrees  $\Delta d = d_{n+1} - d_n$ . The result is

$$\Delta d = 0, 1, 1, 2, 1, 2, 3, 2, 3, 4, 3, 4, 5, 4, 5$$

One remarks that the pattern  $m, m + 1, m, m + 1$ , repeats itself every three steps (and thus has an increment of one unit) leading to linear growth for  $\Delta d$  and thus quadratic for  $d_n$  itself.

Although this result is quite satisfactory as far as the integrability of (1) is concerned, it is somewhat disappointing since the growth is not cubic. A possible explanation can be sought in the difference limit of this mapping. Indeed, by taking the  $q \rightarrow 1$  limit we obtain:

$$x_{n+1} - x_{n-2} = \frac{z_n}{x_n} - \frac{z_{n-1}}{x_{n-1}} \quad (2)$$

where  $z_n = (n + \alpha)/\beta$ . This is just the discrete difference of

$$x_{n+1} + x_n + x_{n-1} = \frac{z_n}{x_n} + 1 \quad (3)$$

The mapping (2) exhibits a quadratic growth, since  $x_n$  is given by the second-order, integrated, form (3). This in turn suggests that the  $q$ -deformed discrete Painlevé I might be integrated to a second-order mapping. This turns out to be possible. We start from the well-known  $q$ -discrete form [19] of the  $P_{34}$  equation:

$$(y_{n+1}y_n - 1)(y_n y_{n-1} - 1) = \zeta_n (y_n - a)(y_n - 1/a) \quad (4)$$

where  $\zeta_n \propto q^{-n}$ . We divide both members by  $y_n \zeta_n$ , upshift the resulting equation and subtract the two, so as to cancel the  $a + 1/a$  term. In the remaining equation we can simplify by a global factor  $(y_n y_{n-1} - 1)$  and we end up with the four-point equation:

$$\frac{y_{n+1}}{\zeta_n} + \frac{1 - 1/\zeta_n}{y_n} = \frac{y_{n-2}}{\zeta_{n-1}} + \frac{1 - 1/\zeta_{n-1}}{y_{n-1}} \quad (5)$$

Next we write  $\zeta_n = q^{-n}/\mu$ ,  $x_n = y_n/(\lambda \zeta_n)$  and obtain equation (1), provided  $\lambda = (1 - q)\beta\mu$  and  $\mu = \alpha(1 - q) - 1$ . Thus (4) is the integrated form of (1) and  $a$  plays the role of the integration constant.

### 3. GAMBIER-LIKE COUPLED MAPPINGS

The other source of integrable third-order mappings are the ones introduced in [20]. The principle is the following. We start from a given integrable second-order mapping for  $x$ :  $x_{n+1} = f(x_n, x_{n-1})$ . (In all the cases analysed in [20] the latter was a simple discrete Painlevé equation). Next, we couple it to a variable  $y$  either through a homographic mapping

$$y_{n+1} = \frac{(\alpha_n x_n + \beta_n) y_n + 1}{y_n + \gamma_n x_n + \delta_n} \quad (6)$$

or a linear one

$$y_{n+1}(\gamma_n x_n + \delta_n) - y_n(\alpha_n x_n + \beta_n) - 1 = 0 \quad (7)$$

In what follows we shall concentrate on the case of the d- $P_1$

$$x_{n+1} + x_{n-1} = \frac{z_n}{x_n} + \frac{1}{x_n^2} \quad (8)$$

where  $z_n = an + b$ . The method used in [20] was based on the singularity

confinement criterion. A first condition for the coupling of (8) with (6) to have confined singularities was (assuming  $\alpha\gamma \neq 0$ ):

$$\beta_n \delta_n = 1 \quad (9a)$$

$$\alpha_n \delta_n + \beta_n \gamma_n = 0 \quad (9b)$$

and the detailed study of this coupling yielded a second condition

$$\beta_{n-1} \beta_n^2 \beta_{n+1} = 1 \quad (10a)$$

$$\frac{\alpha_{n-1}}{\beta_{n-1}} = \frac{\alpha_{n+1}}{\beta_{n+1}} \quad (10b)$$

In the case of linear coupling, only three couplings may, a priori, be compatible with integrability (*i.e.* satisfy a first singularity confinement condition):

$$y_{n+1} = \alpha_n y_n + \frac{1}{\gamma_n x_n} \quad (11a)$$

$$y_{n+1} = \alpha_n x_n y_n + \frac{1}{\delta_n} \quad (11b)$$

$$y_{n+1} = \frac{\beta_n y_n + 1}{\gamma_n x_n} \quad (11c)$$

It turns out that only (11a) can lead to confinement. After the appropriate gauge of  $y$  we have

$$y_{n+1} - y_n = \frac{1}{\gamma_n x_n} \quad (12)$$

and the confinement condition reads  $\gamma_{n+1} - \gamma_{n-1} = 0$ .

At this point an important remark is necessary. While in [20] we have insisted on the confined character of the singularities of the mapping (6) (mapping (8) has by construction confined singularities) the system, as it is explained in [17], should be integrable in all cases. Once the solution of  $x$  is obtained, the integration of the discrete Riccati is given through a straightforward linearisation (and this is even more straightforward for the linear coupling). Still, one would expect the fact that the singularities are confined to confer some special property to the system. In what follows we shall study the degree growth of some initial condition of the d-P<sub>1</sub>, equation (8), coupled to either a discrete Riccati, (6), or a linear mapping, (7). We shall limit ourselves, in order to keep the computations tractable, to autonomous systems which means that the  $z$  in (8) is a free constant. Similarly all the  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$  in (6) and (7) are assumed to be constant. With this assumption the confinement conditions (10)

reduces to  $\beta = \pm 1, \pm i$ ,  $\alpha$  free (and similarly  $\gamma$  free in the case of the coupling (eq. 12)).

Let us start with the following coupling:

$$x_{n+1} + x_{n-1} = \frac{z}{x_n} + \frac{1}{x_n^2} \quad (13a)$$

$$y_{n+1} = \frac{(\alpha x_n + \beta)y_n + 1}{y_n - (\alpha x_n - \beta)/\beta^2} \quad (13b)$$

where (13b) is just (6) where the confinement partial conditions (9) have been implemented. We take as initial conditions  $x_{-1} = r$ ,  $x_0 = p/q$ ,  $y_0 = s$  and assume that the homogeneity degrees of  $r, s$  are 0 and those of  $p, q$  are 1. We iterate the mapping and study the homogeneity degree in  $p, q$  of the numerator/denominator of  $x_n$  and  $y_n$ . For  $x$  we find the following sequence of degrees: 0, 1, 2, 5, 8, 13, 18, 25, 32, 41, 50, ... corresponding to a quadratic growth. We start by assuming  $\beta \neq \pm 1, \pm i$ , *i.e.*, the mapping (13) has unconfined singularities. We obtain the following sequence of degrees:

$$d_n = 0, 1, 3, 8, 16, 29, 47, 72, 104, 145, \dots$$

While this growth appears fast it is still polynomial and indeed *cubic*. This can be easily assessed if one computes the first difference of the degrees  $\Delta d = d_{n+1} - d_n$ . We obtain thus the sequence

$$\Delta d = 1, 2, 5, 8, 13, 18, 25, 32, 41, \dots$$

which is precisely that of the degrees of  $x_n$ . Since the latter grows quadratically the degrees of  $y_n$  grow in a cubic way. As a matter of fact the growth is maximal in the sense that  $d_{y_n} = d_{y_{n-1}} + d_{x_{n-1}}$ , *i.e.*, no simplifications occur from factor cancellation. We turn next to the confining case  $\beta = \pm 1, \pm i$ . We obtain now a slower, quadratic, growth  $d_n = 0, 1, 3, 6, 10, 15, 21, 28, 36, \dots$  *i.e.*  $d_n = n(n+1)/2$ . Thus the confinement conditions lead to simplifications which manage to substantially curb the growth.

Next we examine the coupling (13a) with the two linear mappings

$$y_{n+1} = x_n y_n + 1 \quad (14)$$

and

$$y_{n+1} = y_n + \frac{1}{x_n} \quad (15)$$

In the first case we have a mapping with unconfined singularities. The resulting degree growth for  $y$  is:

$$d_n = 0, 1, 3, 7, 13, 22, 34, 50, 70, 95, 125, \dots$$

Again this is a cubic growth. Computing the first and second ( $\Delta^2 d = d_{n+1} - 2d_n + d_{n-1}$ ) degree-differences we find

$$\Delta d = 1, 2, 4, 6, 9, 12, 16, 20, 25, 30, \dots \quad \text{and} \quad \Delta^2 d = 1, 2, 2, 3, 3, 4, 4, 5, 5, \dots$$

From the linearity of  $\Delta^2 d$  we can confirm that  $d_n$  grows like  $n^3$ , albeit with a *non maximal* growth *i.e.*,  $d_{y_n} < d_{y_{n-1}} + d_{x_{n-1}}$ . This last fact is due to a partial fulfillment of the singularity confinement requirement. Quite expectedly the growth obtained from the coupling (15) is quadratic and it turns out that it is exactly  $d_n = n(n+1)/2$ .

At this stage we can wonder whether from growth arguments one can indeed recover the singularity confinement conditions (as it is done in [7]). In order to investigate this we consider the coupling of (13a) with the linear mapping:

$$y_{n+1} = \frac{y_n(x_n + \beta) + 1}{x_n + \delta} \quad (16)$$

which is indeed just (7) after some scaling and gauge of  $y$ , using the fact that the parameters are constant and under the assumption  $\alpha\gamma \neq 0$ . The study of the degree of  $y$  leads to the following sequence  $d_n = 0, 1, 3, 8, 16, 29, 47, \dots$  *i.e.* a maximal cubic growth. Next we ask how one can obtain a degree smaller than 8 at the fourth iteration. It turns out that the *only* possibility for (16) is  $\beta = \delta = 0$  which means that the mapping assumes the form (15) and the growth becomes quadratic. A similar analysis could in principle be performed in the homographic-coupling case but the bulk of the computations is overwhelming (and the results, given the analysis of the linear coupling, predictable).

In this paper we have presented results on the growth properties of third-order integrable discrete systems, which support our conjecture that a generic integrable  $N$ th-order mapping will have a degree growth like  $n^N$ . Moreover the method we presented here can be easily extended to higher-order systems. As a matter of fact the  $N$ th-order system constructed following this method will be a generalisation of the discrete Gambier system proposed in [17], with the difference that the first equation, instead of being just a discrete Riccati, is a second-order integrable mapping with quadratic growth:

$$\begin{aligned} x_{n+1}^0 &= f(x_n^0, x_{n-1}^0) \\ x_{n+1}^\mu &= \frac{a_\mu(x_n^0, \dots, x_n^{\mu-1})x_n^\mu + b_\mu(x_n^0, \dots, x_n^{\mu-1})}{c_\mu(x_n^0, \dots, x_n^{\mu-1})x_n^\mu + d_\mu(x_n^0, \dots, x_n^{\mu-1})} \quad \mu = 1, \dots, N-2 \end{aligned} \quad (17)$$

Still the approach we presented here is based on partial linearisability. The system is integrated in cascade and the coupling equation is a linear or a linearisable one.

#### 4. CONCLUSIONS

As we have shown the singularity confinement property plays an essential role in the determination of the degree growth. Whenever it is violated (a fact that does not hinder linearisability) the growth is maximal, while when it is satisfied the growth is slower. One can thus wonder what are the growth properties of systems like the higher-order discrete Painlevé equations which satisfy singularity confinement but are not linearisable. Unfortunately not much is known to date about these systems. The only mappings studied are the higher members of some of the discrete Painlevé hierarchies like the d-P<sub>I</sub> and d-P<sub>II</sub> obtained by Creswell and Joshi [21]. The, probably, best-known example of these systems is the string equation [22], higher d-P<sub>I</sub>:

$$x_{n+2} - x_{n-1} = \frac{1}{x_{n+1}} \left( \frac{z_n}{x_n} + c(x_{n+1} + x_n + x_{n-1}) - (x_{n+1} + x_n + x_{n-1})^2 - x_{n-2}x_{n-1} + 1 \right) \quad (18)$$

where  $z_n = an + b$  and  $c$  is a constant. However the study of the degree growth of equation (18) (in a monumental computation by K. Kimura [23]) revealed the quadratic growth

$$d_n = \quad 1, 2, 3, 5, 7, 10, 13, 16, 20, 24, 29, 34, 39, 45, 51, 58, 65, \\ 72, 80, 88, 97, 106, 115, 125, 135, \dots$$

Most probably this behaviour is due to the fact that (18) belongs to a hierarchy and shares many common properties with the “standard” d-P<sub>I</sub>. Clearly the subject of the growth properties of higher-order integrable discrete systems is far from being exhausted.

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