

# EXACT SOLUTIONS OF THE DIRAC EQUATION IN CENTRAL BACKGROUNDS

ION I. COTĂESCU

West University of Timișoara,  
V. Pârvan Ave. 4, 300223 – Timișoara, România

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It is shown that the free Dirac equation in static and spherically symmetric backgrounds of any dimensions can be put in a simple form using a special version of Cartesian gauge in Cartesian coordinates allowing one to separate the generalized spherical variables in terms of angular spinors and a pair of radial equations like in the flat case. In this approach the equation of the free field Dirac in some central backgrounds can be analytically solved obtaining the formula of the energy levels and the corresponding eigenspinors. The example we give are the solutions of the Dirac equation with explicit mass term in  $AdS_{d+1}$  spacetimes and those formed by  $d$ -dimensional spheres with the time trivially added.

## 1. INTRODUCTION

The discovery of the AdS/CFT correspondence [1] brings in a central position the theory of the free fields on the  $(d + 1)$ -dimensional  $AdS$  spacetimes ( $AdS_{d+1}$ ) since their quantum modes can be related to the conformal field theories in the Minkowski-like boundaries of the  $AdS_{d+1}$  manifolds [2]. In this context, the fields with spin are studied in local charts with conformally flat metrics of the Euclidean  $AdS_{d+1}$  backgrounds where their continuous quantum modes are described by simple solutions of the free field equations that can be written as plane waves with specific modulations in string coordinates. In this way the boundary conditions of the regular or irregular modes can be established obtaining the concrete form of the two-point functions of the fields with spin in CFT [3, 4].

In other respects, the *discrete* quantum modes of the free fields in  $AdS_{d+1}$  spacetimes are also of general interest. These can be found in the static local charts of  $AdS_{d+1}$  having the  $SO(d)$ -symmetry, we call here (generalized) central charts. The quantum modes of the scalar free fields in these charts are well-studied either for  $d = 3$  [5] or in the case of any  $d$  [6, 7]. However, the presence of the spin give rise to new difficulties related to the choice of the local orthogonal unholonomic frames (*i.e.*, the gauge fixing) one needs for defining the field components. For this reason, despite of serious investigations in separating

the spherical variables of the Dirac equation in central backgrounds [8], one obtained only several particular solutions in  $AdS$  spacetimes with integration constants whose physical significance remained partially obscure. The complete set of solutions giving the discrete quantum modes of the massive Dirac field in the central charts of the  $AdS_{3+1}$  spacetime was found few years ago [9]. We note that other spherically symmetric solutions of the Dirac equation were obtained in the (3+1)-dimensional de Sitter spacetime [10] and in the (2+1)-dimensional background of the BTZ black hole [11].

Our solutions with central symmetry [9, 10] were derived using a special version of Cartesian tetrad-gauge which points out the central symmetry as a manifest one [12], helping us to simply separate the spherical variables of the Dirac equation in terms of the angular spinors defined in special relativity [13]. Recently, Gu, Ma and Dong have generalized the technique of separating spherical variables of the Dirac equation in Minkowski spacetime to flat (pseudo-Euclidean) spacetimes with generalized spherical coordinates of any dimension [14]. This offered us the opportunity to generalize our method to any  $(d+1)$ -dimensional curved spacetime with static charts [15]. The idea was to write the Dirac equation in a suitable gauge allowing us to take over the whole procedure of the separation of generalized spherical variables worked out for the flat manifolds with central potentials [14]. In this way we obtained simple radial equations that may be solved for particular cases as that of the  $AdS_{d+1}$  spacetimes. Here we would like to present the mentioned previous results and how can be obtained the central modes and the quantization rule of the massive Dirac field in backgrounds  $S^d \times R$  formed as  $d$ -dimensional spheres with the time trivially added.

We start in the second section outlining a convenient version of gauge-covariant Dirac theory in any dimension [15]. In Sec. 3 we define the Cartesian gauge in Cartesian coordinates and we bring the Dirac equation in a simpler form, called *reduced* equation, which has similar properties as that in flat spacetime, including the discrete symmetries. The next section is devoted to the separation of variables in generalized spherical coordinates which generates the radial equations. In Sec. 5 we present the quantum modes of the Dirac field in the central charts with generalized spherical coordinates of the  $AdS_{d+1}$  spacetime, giving the general form of the energy eigenspinors and the energy levels. The energy levels corresponding to the regular modes on  $S^d \times R$  backgrounds are derived in the next section. The parity and charge-conjugation transformations we use are defined in Appendix.

## 2. THE GAUGE-COVARIANT DIRAC THEORY IN $(d+1)$ DIMENSIONS

The theory of the Dirac field in curved spacetimes of any dimensions involves three basic ingredients that can be chosen in different ways. These are

the local chart (*i.e.*, natural frame), the gauge fields defining the local frames and the Clifford algebra. In these conditions, the physical meaning of the whole theory remains independent on the choice of these elements only if we assume that this is gauge-covariant. On the other hand, the generalization to a larger number of dimensions may be physically relevant in the sense of the Kaluza-Klein theories only if the extradimensions are space-like. For this reason we consider here the gauge-covariant theory of the Dirac spinors in backgrounds with  $d + 1$  dimensions among them only one is time-like.

We start with a such pseudo-Riemannian manifold,  $M_{d+1}$ , whose pseudo-Euclidean model has the flat metric  $\eta$  of signature  $(1, d)$ , corresponding to the gauge group  $G(\eta) = SO(1, d)$ . Since the Dirac theory involves local orthogonal (non-holonomic) frames, we choose in  $M_{d+1}$  a local chart with coordinates  $x^\mu$ ,  $\alpha, \dots, \mu, \nu, \dots = 0, 1, 2, \dots, d$ , and introduce local frames using the gauge fields  $e_{\hat{\alpha}}(x)$  and  $\hat{e}^{\hat{\alpha}}(x)$  labeled by local indices (with hat),  $\hat{\alpha}, \dots, \hat{\mu}, \hat{\nu}, \dots = 0, 1, 2, \dots, d$ . Their components accomplish  $e_{\hat{\alpha}}^\mu \hat{e}_{\hat{\nu}}^{\hat{\alpha}} = \delta_{\hat{\nu}}^\mu$ ,  $e_{\hat{\alpha}}^\mu \hat{e}_{\hat{\mu}}^{\hat{\beta}} = \delta_{\hat{\alpha}}^{\hat{\beta}}$ , and  $g_{\mu\nu} e_{\hat{\alpha}}^\mu e_{\hat{\beta}}^\nu = \eta_{\hat{\alpha}\hat{\beta}}$  giving the components of the metric tensor of  $M_{d+1}$  as,  $g_{\mu\nu}(x) = \eta_{\hat{\alpha}\hat{\beta}} \hat{e}_{\hat{\mu}}^{\hat{\alpha}}(x) \hat{e}_{\hat{\nu}}^{\hat{\beta}}(x)$ .

Since the irreducible representations of the Clifford algebra have only an odd number of dimensions, we consider a  $(2m + 1)$ -dimensional Clifford algebra with  $2m \geq d$ , acting on a spinor space with  $2^m$  dimensions. From the basis of this algebra we choose a suitable set of  $d + 1$   $\gamma$ -matrices and match the phase factors in order to have

$$\{\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}\} = 2\eta^{\hat{\alpha}\hat{\beta}}. \quad (1)$$

Then  $(\gamma^0)^\dagger = \gamma^0$  and  $(\gamma^i)^\dagger = -\gamma^i$  ( $i, j, k, \dots = 1, 2, \dots, d$ ) which means that the Dirac adjoint of any matrix  $A$  can be defined in usual manner as  $\bar{A} = \gamma^0 A^\dagger \gamma^0$  so that the  $\gamma$ -matrices be self-adjoint,  $\overline{\gamma^{\hat{\alpha}}} = \gamma^{\hat{\alpha}}$ . Notice that the point-dependent matrices  $\gamma^\mu(x) = e_{\hat{\alpha}}^\mu(x) \gamma^{\hat{\alpha}}$  are also self-adjoint with respect to the Dirac adjoint.

The group  $G(\eta)$  admits an universal covering group  $\tilde{G}(\eta)$  that is simply connected and has the same Lie algebra. The spin operators

$$S^{\hat{\alpha}\hat{\beta}} = \frac{i}{4} [\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}] \quad (2)$$

represent the basis-generators of the *spinor* representation of  $\tilde{G}(\eta)$  which, in general, can be reducible. The real valued parameters of  $\tilde{G}(\eta)$  are the components  $\omega_{\hat{\alpha}\hat{\beta}} = -\omega_{\hat{\beta}\hat{\alpha}}$  of the skew-symmetric tensors,  $\omega$  defining the

operators  $T(\omega) = e^{-iS(\omega)}$ , with  $S(\omega) = \frac{1}{2} \omega_{\hat{\alpha}\hat{\beta}} S^{\hat{\alpha}\hat{\beta}}$ , which transform the gamma-matrices according to the rule

$$[T(\omega)]^{-1} \gamma^{\hat{\alpha}} T(\omega) = \Lambda_{\hat{\beta}}^{\hat{\alpha}}(\omega) \gamma^{\hat{\beta}} \quad (3)$$

where  $\Lambda_{\hat{\beta}}^{\hat{\alpha}}(\omega) = \delta_{\hat{\beta}}^{\hat{\alpha}} + \omega_{\hat{\beta}}^{\hat{\alpha}} + \frac{1}{2} \omega_{\hat{\lambda}}^{\hat{\alpha}} \omega_{\hat{\beta}}^{\hat{\lambda}} + \dots$  is a transformation of  $G(\eta)$ . We specify that the operators  $T(\omega) \in \tilde{G}(\eta)$  are unitary with respect to the Dirac adjoint satisfying  $\overline{T(\omega)} = [T(\omega)]^{-1}$ .

Let us denote by  $\psi$  the Dirac free field of mass  $m$ , and by  $\bar{\psi} = \psi^\dagger \gamma^0$  its Dirac adjoint. In natural units (with  $\hbar = c = 1$ ) its gauge invariant action [17] is

$$S[\psi, e] = \int d^{d+1}x \sqrt{g} \left\{ \frac{i}{2} [\bar{\psi} \gamma^\mu D_\mu \psi - \overline{(D_\mu \psi)} \gamma^\mu \psi] - m \bar{\psi} \psi \right\} \quad (4)$$

where  $g = |\det(g_{\mu\nu})|$  and  $D_\mu = \nabla_\mu + \Gamma_\mu^{spin}$  are the covariant derivatives of the spinor field formed by the usual covariant derivatives  $\nabla_\mu$  (acting in natural indices) and the spin connection  $\Gamma_\mu^{spin} = \frac{i}{2} e_{\hat{\nu}}^\beta (\hat{e}_{\hat{\alpha}}^\mu \Gamma_{\beta\mu}^\alpha - \hat{e}_{\beta,\mu}^{\hat{\sigma}}) S_{\hat{\sigma}}^{\hat{\nu}}$ . The action of these covariant derivatives in the spinor sector is  $D_\mu \psi = (\partial_\mu + \Gamma_\mu^{spin}) \psi$ . The spin connection satisfies  $\bar{\Gamma}_\mu^{spin} = -\Gamma_\mu^{spin}$ , and assures the covariance of the whole theory under the usual gauge transformations [12].

A crucial problem is to find the generators of symmetries at the level of the relativistic quantum mechanics since these operators must commute with the Dirac one. In the absence of other interactions these are produced only by the symmetries of the background. In [12] we defined the external symmetry group  $S(M)$  of a given manifold  $M$  as the universal covering group of the isometry group  $I(M)$ , pointing out that for each matter field there exists a specific representation of  $S(M)$ , *induced* by a linear representation of  $\tilde{G}(\eta)$ , that leaves the field equation invariant. Consequently, the generators of this representation are operators which commute with that of the field equation. They can be calculated starting with the Killing vectors corresponding to the isometries of  $I(M)$  according to a rule obtained by Carter and McLenaghan for the Dirac field [16] which states that for any Killing vector  $k^\mu$  there exists an operator

$$X_k = -ik^\mu D_\mu + \frac{1}{2} k_{\mu;\nu} e_{\hat{\alpha}}^\mu e_{\hat{\beta}}^\nu S^{\hat{\alpha}\hat{\beta}} \quad (5)$$

which commutes with  $\mathcal{E}_D$ . We have shown that these operators are just the generators of the representation of  $S(M)$  in the space of the spinors  $\psi$  that is

induced by the spinor representation of  $\tilde{G}(\eta)$ . Each generator (5) can be divided in an usual orbital part and a spin part, involving the spin matrices, whose forms depend on the choice of the gauge fields. When the spin parts commute with the orbital ones we say that the representation is manifest covariant. These results can be generalized for any  $M_{d+1}$  manifold considering that the operators  $S^{\hat{\alpha}\hat{\beta}}$  are those defined by Eq. (2).

In other respects, from the conservation of the electric charge, one can deduce that when  $e_i^0 = 0$ ,  $i = 1, 2, \dots, d$ , the relativistic scalar product of two spinors [17],

$$\langle \Psi_1, \Psi_2 \rangle = \int_D d^d x \mu(x) \bar{\Psi}_1(x) \gamma^0 \Psi_2(x), \quad (6)$$

defined on the  $d$ -dimensional space domain  $D$  of the chart we use, has the weight function

$$\mu(x) = \sqrt{g(x)} e_0^0(x). \quad (7)$$

The coherence of the whole theory is guaranteed by the fact that the conserved quantity associated to the Killing vector  $k^\mu$ , given by the Noether theorem, is the expectation value  $\langle \Psi, X_k \Psi \rangle$  of the operator (5) calculated using the scalar product (6) [18].

### 3. THE DIRAC EQUATION IN CENTRAL BACKGROUNDS

In what follows we shall focus only on the central manifolds. These have static central charts with Cartesian coordinates, the time  $x^0 = t$  and space Cartesian coordinates  $x^i$ ,  $i = 1, 2, \dots, d$ , where the metric is time-independent and spherically symmetric. The isometry groups of these manifolds,  $I(M_{d+1})$ , include as a subgroup the group of the static central symmetry,  $I_c = T(1) \otimes SO(d)$ , formed by time translations and  $d$ -dimensional orthogonal transformations of the Cartesian space coordinates seen as the components of the vector  $\mathbf{x} \equiv (x^1, x^2, \dots, x^d)$ . The central symmetry requires the metric tensor,  $g(\mathbf{x})$ , to transform manifestly covariant under the linear transformations of the space coordinates,

$$x^i \rightarrow x'^i = R_{ij} x^j, \quad t' = t, \quad (8)$$

produced by any  $R \in SO(d)$ . In these conditions the corresponding line element has the general form

$$ds^2 = g_{\mu\nu}(\mathbf{x}) dx^\mu dx^\nu = A(r) dt^2 - [B(r) \delta_{ij} + C(r) x^i x^j] dx^i dx^j \quad (9)$$

where  $A$ ,  $B$  and  $C$  are arbitrary functions of the Euclidean norm of  $\mathbf{x}$ ,  $r = |\mathbf{x}| = \sqrt{x^i x^i}$ , which is invariant under  $SO(d)$  transformations. In applications it is convenient to replace these functions by new ones,  $u$ ,  $v$  and  $w$ , such that

$$A = w^2, \quad B = \frac{w^2}{v^2}, \quad C = \frac{w^2}{r^2} \left( \frac{1}{u^2} - \frac{1}{v^2} \right). \quad (10)$$

Then the metric appears as the conformal transformation of the simpler one with  $w = 1$ .

In the case of the four-dimensional central backgrounds we proposed a Cartesian gauge in which the gauge fields are static and covariantly transform under transformations similar to (8). We adopt the same type of gauge here, in the Cartesian charts of the central manifolds  $M_{d+1}$ , requiring the 1-forms  $d\hat{x}^{\hat{\alpha}} = \hat{e}_{\mu}^{\hat{\alpha}} dx^{\mu}$  to transform according to the rule  $d\hat{x}^{\hat{\mu}} \rightarrow d\hat{x}'^{\hat{\mu}} = \hat{e}'^{\hat{\mu}}_{\alpha}(x') dx'^{\alpha} = (Rd\hat{x})^{\hat{\mu}}$ . Then the gauge fields must have components of the form

$$\hat{e}_0^0 = \hat{a}(r), \quad \hat{e}_i^0 = \hat{e}_0^i = 0, \quad \hat{e}_j^i = \hat{b}(r)\delta_{ij} + \hat{c}(r)x^i x^j, \quad (11)$$

$$e_0^0 = a(r), \quad e_i^0 = e_0^i = 0, \quad e_j^i = b(r)\delta_{ij} + c(r)x^i x^j, \quad (12)$$

where, according to Eqs. (9) and (10), we find that

$$\hat{a} = w, \quad \hat{b} = \frac{w}{v}, \quad \hat{c} = \frac{1}{r^2} \left( \frac{w}{u} - \frac{w}{v} \right), \quad (13)$$

$$a = \frac{1}{w}, \quad b = \frac{v}{w}, \quad c = \frac{1}{r^2} \left( \frac{u}{w} - \frac{v}{w} \right). \quad (14)$$

Since in this chart

$$\sqrt{g} = B^{\frac{d-1}{2}} [A(B + r^2 C)]^{\frac{1}{2}} = \frac{1}{ab^{d-1}(b + r^2 c)} = \frac{w^{d+1}}{uv^{d-1}}, \quad (15)$$

we obtain the weight function (7) in our Cartesian gauge,

$$\mu = \frac{1}{b^{d-1}(b + r^2 c)} = \frac{w^d}{uv^{d-1}}. \quad (16)$$

From Eqs. (13) and (14) we observe that  $w$  must be positively defined in order to keep the same sense of the time axes of the natural and local frames. In addition, it is convenient to consider that the function  $u$  is positively defined and to try to define the radial coordinate  $r$  so that  $u = 1$ . However, the function  $v$  can be of any sign depending on the orientation of the space axes of the local frame.

The concrete form of the Dirac operator in our gauge can be put in a simpler form introducing the *reduced* Dirac field,  $\tilde{\psi}$ , defined as

$$\Psi(x) = \chi(r)\tilde{\Psi}(x), \quad (17)$$

where

$$\chi = \left[ \sqrt{g(b+r^2c)} \right]^{-1/2} = v^{\frac{d-1}{2}} w^{-\frac{d}{2}}. \quad (18)$$

After this substitution we obtain the reduced Dirac equation,  $\tilde{\mathcal{E}}_D \tilde{\Psi} = m\tilde{\Psi}$ , in which the reduced Dirac operator,

$$\tilde{\mathcal{E}}_D = ia(r)\gamma^0\partial_t + ib(r)\gamma^i\partial_i + ic(r)(\gamma^i x^i) \left( \frac{d-1}{2} + x^i\partial_i \right), \quad (19)$$

is independent on the derivatives of  $a$ ,  $b$  and  $c$ .

We observe that the reduced Dirac equation has a manifest symmetry similar to that of the Dirac equation in flat backgrounds with central potentials. This is the consequence of our version of Cartesian gauge which produces usual linear representation of the external symmetry group  $S_c(M_{d+1})$ , associated with  $I_c(M_{d+1})$ , showing off the central symmetry in a manifest covariant form.

Indeed, taking into account that  $S_c(M_{d+1}) = T(1) \otimes \tilde{SO}(d)$  involves the group  $\tilde{SO}(d) \subset \tilde{G}(\eta)$  which is the universal covering group of  $SO(d)$ , we conclude that the spin operators  $S^{ij}$  defined by Eq. (2) are the basis-generators of the spinor representation of  $\tilde{SO}(d)$ . Furthermore, following the method of Ref. [12], we find that the basis-generators of the induced representation of  $S_c(M_{d+1})$  given by Eq. (5) are the  $T(1)$  generator,  $i\partial_t$ , and the  $\tilde{SO}(d)$  ones,

$$\mathcal{J}_{ij} = L_{ij} + S_{ij} = -i(x^i\partial_j - x^j\partial_i) + S_{ij}, \quad (20)$$

which play the role of total angular momentum. They commute with the operator (19) and, obviously, have a manifest covariant form.

On the other hand, the reduced Dirac equation can be put in Hamiltonian form  $i\partial_t\tilde{\Psi} = \mathcal{H}\tilde{\Psi}$  as in the case of the flat manifolds using the operators [15]

$$\mathcal{H} = -i\frac{u(r)}{r^2}(\gamma^0\gamma^i x^i) \left( \frac{d-1}{2} + x^i\partial_i \right) - i\frac{v(r)}{r^2}(\gamma^i x^i)\mathcal{K} + w(r)\gamma^0 m, \quad (21)$$

$$\mathcal{K} = \gamma^0 \left( S^{ij}L_{ij} + \frac{d-1}{2} \right). \quad (22)$$

Since  $\gamma^0$  commutes with  $\mathcal{J}_{ij}$  and  $\mathcal{K}$  we have

$$[\mathcal{H}, \mathcal{J}_{ij}] = 0, \quad [\mathcal{H}, \mathcal{K}] = 0. \quad (23)$$

Consequently, all the properties related to the conservation of the angular momentum, including the separation of variables in spherical coordinates, will be similar to those of the usual Dirac theory in the flat spacetimes with  $d + 1$  dimensions [14]. Moreover, we note that the discrete transformations of parity and charge conjugation can be also defined. These leave Eq. (19) invariant and have the same significance as in special relativity [19, 13]. Thus, for example, the charge conjugation transforms each particular solution of positive frequency into the corresponding one of negative frequency (see the Appendix).

#### 4. SEPARATION OF SPHERICAL COORDINATES

The next step is to introduce the generalized spherical coordinates,  $r, \theta_1, \theta_2, \dots, \theta_{d-1}$ , [20] associated with the space coordinates of our natural Cartesian frame,

$$\begin{aligned} x^1 &= r \cos \theta_1 \sin \theta_2 \dots \sin \theta_{d-1}, \\ x^2 &= r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{d-1}, \\ &\vdots \\ x^d &= r \cos \theta_{d-1}. \end{aligned} \tag{24}$$

Then from Eqs. (9) and (10) one obtains the line element

$$ds^2 = w^2 dt^2 - \frac{w^2}{u^2} dr^2 - \frac{w^2}{v^2} r^2 d\theta^2, \tag{25}$$

where  $d\theta^2$  is the usual line element on the sphere  $S^{d-1}$  [20]. Consequently, in this chart we have

$$\sqrt{g_s} = \sqrt{g} r^{d-1} \prod_{a=1}^{d-1} (\sin \theta_a)^{a-1}, \tag{26}$$

and similarly for the weight function (7). The special form of the reduced Dirac equation allows one to separate the spherical variables as in the case of the central motion in flat spacetimes, using the angular spinors  $\phi_{\kappa, (j)}(\hat{\mathbf{x}})$  defined in Ref. [14]. These depend only on the spherical variables represented by the unit vector  $\hat{\mathbf{x}} = \mathbf{x}/r$ , being determined by a set of weights  $(j)$  of an irreducible representation of  $\tilde{SO}(d)$  and the eigenvalue  $\kappa = \pm |\kappa|$  of the operator  $\mathcal{K}$  which concentrates all the angular operators of  $\mathcal{H}$ .

The separation procedure is complicated being different for even or odd values of  $d$  [14]. For odd  $d$  the particular solution of given energy,  $E$ , and



positive frequency corresponding to the irreducible representation ( $j$ ) may have the form

$$\tilde{\Psi}_{E,\kappa,(j)}(t, \mathbf{x}) = r^{-\frac{d-1}{2}} e^{-iEt} \begin{pmatrix} f_{E,\kappa}^+(r) \phi_{\kappa,(j)}(\hat{\mathbf{x}}) \\ if_{E,\kappa}^-(r) \phi_{-\kappa,(j)}(\hat{\mathbf{x}}) \end{pmatrix}, \quad (27)$$

where  $\kappa$  can take positive or negative values. However, if  $d$  is even then there are pairs of associated irreducible representations, ( $j_1$ ) and ( $j_2$ ), giving the same eigenvalue of the first Casimir operator, among them one takes the first one for positive values of  $\kappa$  and the second one for the negative values,  $\kappa = -|\kappa|$ . Consequently, the particular solutions read

$$\begin{aligned} \tilde{\Psi}_{E,|\kappa|,(j_1)}(t, \mathbf{x}) &= r^{-\frac{d-1}{2}} e^{-iEt} \times \\ &\times \left[ f_{E,|\kappa|}^+(r) \phi_{|\kappa|,(j_1)}(\hat{\mathbf{x}}) + if_{E,-|\kappa|}^-(r) \phi_{-|\kappa|,(j_1)}(\hat{\mathbf{x}}) \right], \end{aligned} \quad (28)$$

$$\begin{aligned} \tilde{\Psi}_{E,-|\kappa|,(j_2)}(t, \mathbf{x}) &= r^{-\frac{d-1}{2}} e^{-iEt} \times \\ &\times \left[ f_{E,-|\kappa|}^+(r) \phi_{-|\kappa|,(j_2)}(\hat{\mathbf{x}}) + if_{E,|\kappa|}^-(r) \phi_{|\kappa|,(j_2)}(\hat{\mathbf{x}}) \right]. \end{aligned} \quad (29)$$

It is remarkable that both these types of solutions involve the same set of radial wave functions that depend on  $E$  and

$$\kappa = \pm \left( \frac{d-1}{2} + l \right), \quad l = 0, 1, 2, \dots, \quad (30)$$

where  $l$  is an auxiliary orbital quantum number defining the representations ( $j$ ), ( $j_1$ ) and ( $j_2$ ) [14]. The radial wave functions obey a pair of radial equations similar to those found in the case of  $d=3$  [9]. For a given  $\kappa$ , we embed these radial equations in the eigenvalue problem

$$H_\kappa f_{E,\kappa} = E f_{E,\kappa} \quad (31)$$

of the *radial* Hamiltonian operator

$$H_\kappa = \begin{pmatrix} mw & u \frac{d}{dr} + \kappa \frac{v}{r} \\ -u \frac{d}{dr} + \kappa \frac{v}{r} & -mw \end{pmatrix} \quad (32)$$

acting in the two-dimensional space with elements  $f = (f^+, f^-)^T$ . After the separation of variables the scalar product (6) splits in angular and radial terms. Supposing that the angular spinors are normalized with respect to an angular scalar product, we remain with the radial scalar product

$$\langle \Psi_1, \Psi_2 \rangle = \langle f_1, f_2 \rangle = \int_{D_r} \frac{dr}{u} f_1^\dagger f_2. \quad (33)$$

The radial weight function  $\mu\chi^2 = 1/u$ , resulted from Eqs. (16) and (18), is just that we need in order to have  $(u\partial_r)^\dagger = -u\partial_r$ , such that  $H_\kappa$  be Hermitian with respect to the scalar product (33). Thus we obtain an independent radial problem which has to be solved in each concrete case separately using appropriate methods.

Let us observe that for  $m=0$  the Hamiltonian operator of the radial problem does not involve explicitly the function  $w$ . This means that the massless Dirac field produces the same energy spectrum in charts whose metrics differ to each other through conformal transformations. Thus we recover a well-known result of the theory of conformal transformations [17] in the particular case of the central backgrounds we study here.

### 5. QUANTUM MODES IN $AdS_{d+1}$ SPACETIMES

The  $AdS_{d+1}$  spacetime is the hyperboloid  $\eta_{AB}Z^AZ^B = R^2$  of radius  $R=1/\omega$  in the  $(d+2)$ -dimensional flat spacetime of coordinates  $Z^{-1}, Z^0, Z^1, \dots, Z^d$  and metric  $\eta_{AB} = \text{diag}(1, 1, -1, \dots, -1)$ ,  $A, B = -1, 0, 1, \dots, d$ . Here we consider the static chart of Cartesian coordinates  $(t, \mathbf{x})$  defined as

$$\begin{aligned} Z^{-1} &= R \sec \omega r \cos \omega t, \\ Z^0 &= R \sec \omega r \sin \omega t, \\ \mathbf{Z} &= R \frac{\mathbf{x}}{r} \tan \omega r, \end{aligned} \quad (34)$$

and the corresponding static chart with generalized spherical coordinates (24) where the line element reads [5, 6]

$$ds^2 = \eta_{AB}dZ^AdZ^B = \sec^2 \omega r \left( dt^2 - dr^2 - \frac{1}{\omega^2} \sin^2 \omega r d\theta^2 \right). \quad (35)$$

In this chart  $r \in D_r = [0, \pi/2\omega)$  and, therefore, the whole space domain is  $D = D_r \times S^{d-1}$ . In addition, we specify that the time of  $AdS_{d+1}$  must satisfy  $t \in [-\pi/\omega, \pi/\omega)$  while  $t \in (-\infty, \infty)$  defines the universal covering spacetime of  $AdS_{d+1}$  ( $CAdS_{d+1}$ ) [5].

Furthermore, from Eq. (35) we identify

$$u(r) = 1, \quad w(r) = \sec \omega r, \quad v(r) = \omega r \csc \omega r, \quad (36)$$

and, introducing the notation  $k = m/\omega$  (*i.e.*  $mc^2/\hbar\omega$  in usual units), we find the Hamiltonian of the radial problem

$$H_{\kappa} = \begin{pmatrix} \omega k \sec \omega r & \frac{d}{dr} + \omega \kappa \csc \omega r \\ -\frac{d}{dr} + \omega \kappa \csc \omega r & -\omega k \sec \omega r \end{pmatrix}. \quad (37)$$

Its form suggests us to perform the local rotation  $f \rightarrow \hat{f} = U(r)f = (\hat{f}^+, \hat{f}^-)^T$  where

$$U(r) = \begin{pmatrix} \cos \frac{\omega r}{2} & \sin \frac{\omega r}{2} \\ -\sin \frac{\omega r}{2} & \cos \frac{\omega r}{2} \end{pmatrix}. \quad (38)$$

The transformed Hamiltonian,

$$\hat{H}_{\kappa} = U(r)H_{\kappa}U^{\dagger}(r) - \frac{\omega}{2}1_{2 \times 2} = \begin{pmatrix} v & \frac{d}{dr} + W_{\kappa} \\ -\frac{d}{dr} + W_{\kappa} & -v \end{pmatrix} \quad (39)$$

giving the new eigenvalue problem

$$\hat{H}_{\kappa}\hat{f} = \left(E - \frac{\omega}{2}\right)\hat{f}, \quad (40)$$

has supersymmetry since it has the requested specific form with  $v = \omega(k + \kappa)$  and the superpotential

$$W_{\kappa}(r) = \omega(\kappa \cot \omega r - k \tan \omega r). \quad (41)$$

Consequently, the transformed radial wave functions,  $\hat{f}^{\pm}$ , satisfy the second order equations

$$\left(-\frac{1}{\omega^2} \frac{d^2}{dr^2} + \frac{k(k \mp 1)}{\cos^2 \omega r} + \frac{\kappa(\kappa \mp 1)}{\sin^2 \omega r}\right)\hat{f}^{(\pm)}(r) = \epsilon^2 \hat{f}^{(\pm)}(r), \quad (42)$$

where  $\epsilon = E/\omega - 1/2$ . The general solutions can be written in terms of Gauss hypergeometric functions [21],

$$\begin{aligned} \hat{f}^{\pm}(r) &= N_{\pm} \sin^{2s_{\pm}} \omega r \cos^{2p_{\pm}} \omega r \\ &\times F\left(s_{\pm} + p_{\pm} - \frac{\epsilon}{2}, s_{\pm} + p_{\pm} + \frac{\epsilon}{2}, 2s_{\pm} + \frac{1}{2}, \sin^2 \omega r\right), \end{aligned} \quad (43)$$

depending on parameters that must satisfy

$$2s_{\pm}(2s_{\pm} - 1) = \kappa(\kappa \mp 1), \quad (44)$$

$$2p_{\pm}(2p_{\pm} - 1) = k(k \mp 1), \quad (45)$$

and on the normalization factors  $N_{\pm}$ . The next step is to select the suitable values of these parameters and to calculate  $N_{+}/N_{-}$  such that the functions  $\hat{f}^{\pm}$  should be solutions of the transformed radial problem (40) with a good physical meaning. This can be achieved only when  $F$  is a polynomial selected by a suitable quantization condition since otherwise  $F$  is strongly divergent for  $\sin^2 \omega r \rightarrow 1$ . Then the functions  $\hat{f}^{\pm}$  will be square integrable with normalization factors calculated according to the condition

$$\langle f, f \rangle = \langle \hat{f}, \hat{f} \rangle = \int_{D_r} dr (|\hat{f}^{+}(r)|^2 + |\hat{f}^{-}(r)|^2) = 1, \quad (46)$$

resulted from the fact that the matrix (38) is orthogonal.

The discrete energy spectrum is given by the particle-like *CAdS* quantization conditions

$$\epsilon = 2(n_{\pm} + s_{\pm} + p_{\pm}), \quad \epsilon > 0, \quad (47)$$

that must be compatible with each other, *i.e.*

$$n_{+} + s_{+} + p_{+} = n_{-} + s_{-} + p_{-}. \quad (48)$$

Hereby we see that there is only one independent *radial* quantum number,  $n_r = 0, 1, 2, \dots$ . In other respects, if we express the solutions (43) in terms of Jacobi polynomials, we observe that these functions remain square integrable for  $2s_{\pm} > -1/2$  and  $2p_{\pm} > -1/2$ . Since from Eq. (30) we have  $|\kappa| \geq \frac{d-1}{2}$ , we are forced to consider only the positive solutions of Eqs. (44). However, the Eqs. (45) admit either the solutions  $2p_{+} = k$  and  $2p_{-} = k+1$  defining the boundary conditions of *regular* modes or other possible values,  $2p_{+} = -k+1$  and  $2p_{-} = -k$ , giving the *irregular* modes for which  $k < 1/2$  [9]. Here we restrict ourselves to present only the energy eigenspinors of the regular modes on *CAdS* [15].

Let us consider first  $\kappa = |\kappa|$  and choose  $2s_{+} = \kappa$  and  $2s_{-} = \kappa + 1$  for which Eq. (48) is accomplished provided  $n_{+} = n_r$  and  $n_{-} = n_r - 1$ . Then the functions  $\hat{f}^{\pm}$  given by Eqs. (43) will represent a good solution of the transformed radial problem (40) for suitable values of the normalization factors. For  $\kappa = -|\kappa|$  we use the same procedure finding  $2s_{+} = |\kappa| + 1$ ,  $2s_{-} = |\kappa|$ , and  $n_{+} = n_{-} = n_r$  in the case of the regular modes. The explicit form of the normalized radial functions is given in Ref. [15]. Finally, using the inverse transformation of (38) we obtain the radial wave functions

$$f_{n_r, \kappa}^{\pm}(r) = \hat{f}_{n_r, \kappa}^{\pm}(r) \cos \frac{\omega r}{2} \mp \hat{f}_{n_r, \kappa}^{\mp}(r) \sin \frac{\omega r}{2} \quad (49)$$

of the particular solutions of positive frequency (27), (28) and (29) of the reduced Dirac equation. The last step is to restore the form of the field  $\psi$ , according to Eqs. (17), (18) and (36), writing down the particular solutions of positive frequency of the original Dirac equation,

$$\Psi_{n_r, \kappa, (j)}^{pa}(t, \mathbf{x}) = \left( \frac{\omega r}{\sin \omega r} \right)^{\frac{d-1}{2}} \cos^{\frac{d}{2}} \omega r \tilde{\Psi}_{n_r, \kappa, (j)}(t, \mathbf{x}). \quad (50)$$

If these are interpreted as particle-like solutions then the antiparticle-like ones can be derived directly through the charge conjugation as

$$\Psi_{n_r, \kappa, (j)}^{ap} = \left( \Psi_{n_r, \kappa, (j)}^{pa} \right)^c = C \left( \bar{\Psi}_{n_r, \kappa, (j)}^{pa} \right)^T, \quad (51)$$

where the matrix  $C$  is defined by Eq. (68).

The energy levels result from Eq. (47) where we must take into account that  $\omega k = m$  and  $\omega \epsilon = E - \omega/2$ . Thus we obtain

$$E_{n_r, \kappa} = \begin{cases} m + \omega \left( 2n_r + \kappa + \frac{1}{2} \right) & \text{for } \kappa = |\kappa| \\ m + \omega \left( 2n_r + |\kappa| + \frac{3}{2} \right) & \text{for } \kappa = -|\kappa| \end{cases} \quad (52)$$

which suggests us to introduce the principal quantum number

$$n = \begin{cases} 2n_r + |\kappa| - \frac{d-1}{2} = 2n_r + l & \text{for } \kappa = |\kappa| \\ 2n_r + |\kappa| - \frac{d-1}{2} + 1 = 2n_r + l + 1 & \text{for } \kappa = -|\kappa| \end{cases} \quad (53)$$

taking the values 0, 1, 2, ... since  $l$  ranges as in Eq. (30) and  $n_r = 0, 1, 2, \dots$ . With its help we can write the compact formula of the energy levels [15]

$$E_n = m + \omega \left( n + \frac{d}{2} \right), \quad n = 0, 1, 2, \dots \quad (54)$$

which is similar to that of the  $d$ -dimensional homogeneous harmonic oscillator but having a relativistic rest energy. These energy levels are deeply degenerated and the problem of determining the degree of degeneracy seems to be complicated requiring a special study.

## 6. QUANTUM MODES IN $S^d \times R$ SPACETIMES

Let us consider now the central charts of the  $S^d \times R$  backgrounds where the line element has the form

$$ds^2 = dt^2 - dr^2 - \frac{1}{\omega^2} \sin^2 \omega r d\theta^2, \quad (55)$$

being in fact a conformal transformation of  $AdS_{d+1}$  metric (35) having a similar space domain  $\hat{D} = \hat{D}_r \times S^{d-1}$ , with the same  $S^{d-1}$  spherical part but with a larger radial domain  $\hat{D}_r = [0, \pi/\omega)$ . Hereby we identify

$$u(r) = 1, \quad w(r) = 1, \quad v(r) = \omega r \csc \omega r, \quad (56)$$

and, keeping the notation  $k = m/\omega$ , we obtain the radial Hamiltonian

$$H_\kappa = \begin{pmatrix} \omega k & \frac{d}{dr} + \omega \kappa \csc \omega r \\ -\frac{d}{dr} + \omega \kappa \csc \omega r & -\omega k \end{pmatrix}, \quad (57)$$

which has a manifest supersymmetry. Consequently, the radial wave functions  $f^\pm$  of the eigenvalue problem (31) satisfy the second order equations

$$\left( -\frac{1}{\omega^2} \frac{d^2}{dr^2} + \frac{\kappa^2 \mp \kappa \cos \omega r}{\sin^2 \omega r} \right) f^{(\pm)}(r) = \epsilon^2 f^{(\pm)}(r), \quad (58)$$

where  $\epsilon = E/\omega$ . Furthermore, we denote  $\hat{\epsilon}^2 = 4(\epsilon^2 - k^2)$  and introduce the new variable  $\rho = \frac{1}{2}\omega r$  obtaining the familiar form of the above second order equations,

$$\left( -\frac{d^2}{d\rho^2} + \frac{\kappa(\kappa \pm 1)}{\cos^2 \rho} + \frac{\kappa(\kappa \mp 1)}{\sin^2 \rho} \right) f^{(\pm)}(\rho) = \hat{\epsilon}^2 f^{(\pm)}(\rho), \quad (59)$$

that have the particular solutions

$$f^\pm(\rho) = \hat{N}_\pm \sin^{2s_\pm} \rho \cos^{2p_\pm} \rho \times \\ \times F\left(s_\pm + p_\pm - \frac{\hat{\epsilon}}{2}, s_\pm + p_\pm + \frac{\hat{\epsilon}}{2}, 2s_\pm + \frac{1}{2}, \sin^2 \rho\right). \quad (60)$$

These depend on the parameters  $s_\pm$  and  $p_\pm$  that satisfy

$$2s_\pm(2s_\pm - 1) = \kappa(\kappa \mp 1), \quad (61)$$

$$2p_\pm(2p_\pm - 1) = \kappa(\kappa \pm 1), \quad (62)$$

and on the new normalization factors  $\hat{N}_\pm$ .

As in the previous case the regular quantum modes are correctly defined by square integrable radial functions which correspond only to the positive

solutions of the above equations. Thus for  $\kappa = |\kappa|$  we must choose  $2s_+ = \kappa$ ,  $2s_- = \kappa + 1$ ,  $2p_+ = \kappa + 1$  and  $2p_- = \kappa$  while when  $\kappa = -|\kappa|$  we have to take  $2s_+ = \kappa + 1$ ,  $2s_- = \kappa$ ,  $2p_+ = \kappa$  and  $2p_- = \kappa + 1$ . It is remarkable that in both these cases we have

$$2s_+ + 2p_+ = 2s_- + 2p_- = 2|\kappa| + 1. \quad (63)$$

Furthermore, in order to avoid the divergence of these functions for  $\sin^2 \rho \rightarrow 1$  we are forced to impose the quantization condition

$$s_{\pm} + p_{\pm} - \frac{1}{2}\hat{\epsilon} = -\hat{n}_r, \quad (64)$$

where  $\hat{n}_r$  is a new radial quantum number taking the values  $\hat{n}_r = 0, 1, 2, \dots$ . With its help we can define the principal quantum number  $n = \hat{n}_r + l$  finding that the energy levels,  $E_n$ , are given by

$$(E_n)^2 = m^2 + \omega^2 \left( n + \frac{d}{2} \right)^2, \quad n = 0, 1, 2, \dots \quad (65)$$

We observed that the energy spectra of the massless Dirac field remain unchanged under conformal transformations. Therefore, a large class of radial problems with the same equidistant spectrum,

$$E_n^0 = \omega \left( n + \frac{d}{2} \right), \quad n = 0, 1, 2, \dots, \quad (66)$$

is generated by the massless Dirac field in different central backgrounds whose metrics are conformal transformations of the  $AdS_{d+1}$  central one, *i.e.* metrics of the form (25) with  $u(r) = 1$ ,  $v(r) = \omega r \csc \omega r$  and arbitrary  $w(r)$ . This explains why for  $m = 0$  the formulas (54) and (65) lead to the same energy spectrum (66). We note that this result is in accordance with those found in Ref. [22].

## 7. COMMENTS

We demonstrated that our version of Cartesian gauge in central charts leads to simple reduced Dirac equations in Cartesian coordinates and similar radial problems in spherical coordinates for any central background  $M_{d+1}$ . These reduce to a pair of radial equations that depend on  $d$  only through the quantum number  $\kappa$ . This property may allow one to solve the radial problem for large sets of spacetimes of the same type and to study how depend the quantum modes on  $d$ .

In the case of the  $CAdS_{d+1}$  spacetimes we found that the energy spectrum of the massive Dirac field minimally coupled with gravity is discrete and

equidistant having the ground state energy  $E_0 = \omega\left(k + \frac{d}{2}\right)$ . This result is in accordance with the scaling dimension of the spinor operator of CFT which in our notations reads  $\Delta = k + \frac{d}{2}$  [3].

We must specify that equidistant energy spectra appear in many other problems with central symmetry. We remind the reader that the scalar field on  $CAAdS_{d+1}$  has also equidistant energy levels, [6]

$$E_n^{sc} = \sqrt{m^2 + \frac{d^2}{4}\omega^2} + \omega\left(n + \frac{d}{2}\right), \quad n = 0, 1, 2, \dots, \quad (67)$$

with the same energy quanta as in Eq. (54) but having another ground state energy. This difference is due to the fact that in  $AdS_{d+1}$  spacetimes the free Dirac equation is no longer the square root of the Klein-Gordon equation with the same mass.

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## APPENDIX

### PARITY AND CHARGE CONJUGATION

Let us assume that there are  $s+1$  symmetric  $\gamma$ -matrices, namely  $\gamma^0$  and  $s$  matrices with space-like indices,  $\gamma^{\hat{\alpha}_1}, \gamma^{\hat{\alpha}_2}, \dots, \gamma^{\hat{\alpha}_s}$ . Then the matrix [15]

$$C = (-1)^{\frac{s}{2}} \gamma^0 \gamma^{\hat{\alpha}_1} \gamma^{\hat{\alpha}_2} \dots \gamma^{\hat{\alpha}_s} \quad (68)$$

has the properties  $C^{-1} = C^T = (-1)^s \bar{C}$  and  $C\gamma^{\hat{\mu}}C^{-1} = (-1)^s (\gamma^{\hat{\mu}})^T$ . With its help one can define the charge conjugated spinor  $\psi^c = C\bar{\psi}^T$  of the spinor  $\psi$  and verify that the reduced Dirac equation given by the operator (19) remains invariant under the charge conjugation  $\psi \rightarrow \psi^c$ . Note that this transformation is point-independent which suggests that the vacuum state could be stable (or invariant [17]) in quantum field theories based on field equations invariant under this type of charge-conjugation. Particularly, it is known that the Euclidean vacuum of the  $AdS$  spacetime is invariant [17].

## REFERENCES

1. J. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998).
2. S. S. Guber, I. R. Klebanov and A. M. Polyakov, *Phys. Lett.* **B428**, 105 (1998); E. Witten, *Adv. Theor. Math. Phys.* **2**, 253 (1998).



3. M. Henningson and K. Sfetsos, *Phys. Lett.* **B431**, 63 (1998).
4. W. Münk and K. S. Viswanathan, *Phys. Rev. D* **58**, 106006 (1998); R. C. Rashkov, *Phys. Lett.* **B466**, 190 (1999).
5. S. J. Avis, C. J. Isham and D. Storey, *Phys. Rev. D* **18**, 3565 (1978); P. Breitenlohner and D. Z. Freedman, *Ann. Phys. (NY)* **144**, 249 (1982) D. J. Navarro and J. Navarro-Salas, *J. Math. Phys.* **37**, 6006 (1996).
6. C. P. Burgess and C. A. Lütken, *Phys. Lett.* **153B**, 137 (1985).
7. I. I. Cotăescu, *Phys. Rev. D* **60**, 107504 (1999).
8. V. S. Otchik, *Class. Quantum Grav.* **2**, 539 (1985); G. V. Shishkin, *Class. Quantum Grav.* **8**, 175 (1991); G. V. Shishkin and V. M. Villalba, *J. Math. Phys.* **30**, 2132 (1989); G. V. Shishkin and V. M. Villalba, *J. Math. Phys.* **33**, 2093 (1992); V. M. Villalba, *Mod. Phys. Lett. A* **8**, 2351 (1993).
9. I. I. Cotăescu, *Mod. Phys. Lett. A* **13**, 2923 (1998); I. I. Cotăescu, *Phys. Rev. D* **60**, 124006 (1999).
10. I. I. Cotăescu, *Mod. Phys. Lett. A* **13**, 2991 (1998).
11. V. Cardoso and J. P. S. Lemos, *Phys. Rev. D* **63**, 124015 (2001).
12. I. I. Cotăescu, *J. Phys. A: Math. Gen.* **33**, 1977 (2000).
13. B. Thaller, *The Dirac Equation*, Springer Verlag, Berlin, Heidelberg (1992).
14. X.-Y. Gu, Z.-Q. Ma and S.-H. Dong, *Int. J. Mod. Phys. E* **33**, 335 (2002); physics/0209039
15. I. I. Cotăescu, *Int. J. Mod. Phys. A* **19**, 2217 (2004).
16. B. Carter and R. G. McLenaghan, *Phys. Rev. D* **19**, 1093 (1979).
17. N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge (1982).
18. I. I. Cotăescu, *Phys. Rev. D* **65**, 084008 (2002).
19. S. Weinberg, *The Quantum Theory of Fields*, Univ. Press, Cambridge (1995).
20. A. Erdelyi, *Higher Transcendental Functions*, McGraw-Hill, New York (1953); M. E. Taylor, *Partial Differential Equations*, Springer Verlag, New York (1996).
21. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover (1964).
22. R. Camporesi and A. Higuchi, *J. Math. Phys.* **35**, 4217 (1994).