A HOLOMORPHIC REPRESENTATION OF LIE ALGEBRAS SEMIDIRECT SUM OF SEMISIMPLE AND HEISENBERG ALGEBRAS

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The algebra semidirect sum of the real three-dimensional Heisenberg algebra and the $\mathfrak{su}(1, 1)$ algebra admits a realization by first order differential operators with polynomial coefficients. In order to construct such a representation we use coherent state vectors based on the Kähler manifold which as set is the product of the complex plane and the unit disk. We present the Hilbert space of holomorphic functions on which the differential operators act: the reproducing kernel, the group invariant measure, the base of orthonormal polynomials.

1. INTRODUCTION

In this paper we construct representations of Lie algebras which are semidirect sum of Heisenberg algebras and semisimple Lie algebras by first order differential operators with holomorphic polynomials coefficients. The natural framework for such an approach is furnished by the coherent state (CS)-groups, *i.e.*, groups which admit an orbit which is a complex submanifold of a projective Hilbert space [15, 16]. Such type of groups contains the compact groups, the simple hermitian groups, certain solvable groups and also some mixed groups as the semidirect product of the Heisenberg-Weyl (HW) group and the symplectic group [16].

For hermitian symmetric spaces we have produced [5, 6] simple formulas which show that the differential action of the generators of a hermitian group G on holomorphic functions defined on the hermitian symmetric spaces G/H can be written down as a sum of two terms, one a polynomial P, and the second one a sum of partial derivatives times some polynomials Q-s, the degree of polynomials being less than 3. We have generalized the results of [5, 6] to Kähler CS-orbits of semisimple Lie groups [7, 9]. The differential action of the generators of the groups is of the same type as in the case of hermitian symmetric orbits, *i.e.*, first order differential operators with holomorphic polynomial coefficients, but the maximal degree of the polynomials is grater than 2. In [8] we have discussed the hypothesis that *the generators of CS-groups admit representations by first order differential operators with holomorphic polynomials coefficients on CS-manifolds*. We have obtained [11] such a representation for the so called *Jacobi algebra* (cf. p. 178 in [16]), *i.e.* the Lie algebra semidirect sum of the three-dimensional Heisenberg algebra $\mathfrak{h}_1$ and the algebra of the group $SU(1, 1)$, $\mathfrak{g} := \mathfrak{h}_1 \rtimes \mathfrak{su}(1,1)$.

The paper is laid out as follows. §2 presents the Jacobi algebra. Perelomov's CS-vectors [18] associated with the Jacobi group $G := HW \rtimes SU(1,1)$ (cf. denomination used at p. 701 in [16]) are based on the manifold $M := HW/\mathbb{R} \times SU(1,1)/U(1) = \mathbb{C} \times \mathcal{D}_1$. Lemma 1 expresses the differential action of the generators of the Jacobi group. In Lemma 2 of §3 we calculate the reproducing kernel $K : M \times \bar{M} \to \mathbb{C}$. Several facts concerning the representations of the groups HW and $SU(1, 1)$ are collected in §4. As a consequence of the fact that the Heisenberg algebra is an ideal of the Jacobi algebra, we obtain several relations for the representations of the corresponding groups and we find the recipe to change the order of the representations of the groups HW and $SU(1, 1)$. Proposition 1 expresses the action of the Jacobi group on Perelomov's CS-vectors. In §5 we construct the symmetric Fock space attached to the reproducing kernel K from the symmetric Fock spaces associated with the groups HW and $SU(1, 1)$. The G-invariant Kähler two-form ω and the volume form on the manifold M are calculated using the general prescription of [7, 8]. A simple application [6, 7] to equations of motion [12] on M determined by linear Hamiltonians in the generators of the Jacobi group are presented at the end. More details about the proofs can be found in [11]. Some of the relations presented in this paper have appeared earlier in the context of squeezed states [14] in quantum optics [19].

2. THE DIFFERENTIAL ACTION OF THE JACOBI ALGEBRA

The Heisenberg-Weyl group is the group with the 3-dimensional real Lie algebra isomorphic to the Heisenberg algebra

$$\mathfrak{h}_1 \equiv \mathfrak{g}_{HW} = < is1 + xa^+ - \bar{x}a >_{s\in\mathbb{R},x\in\mathbb{C}}, \qquad (2.1)$$

where a^+ (a) are the boson creation (respectively, annihilation) operators which verify the canonical commutation relations (CCR) (2.5a).

We consider the Lie algebra of the group $SU(1, 1)$:

$$\mathfrak{su}(1,1) = < 2i\theta K_0 + yK_+ - \bar{y}K_- >_{\theta\in\mathbb{R},y\in\mathbb{C}}, \qquad (2.2)$$

where the generators $K_{0,+,-}$ verify the standard commutation relations (2.5b). We consider the following matrix realization of the algebra $\mathfrak{su}(1,1)$:

$$K_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K_+ = i\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad K_- = i\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.3}$$

The Jacobi algebra is the semidirect sum

$$\mathfrak{g} := \mathfrak{h}_1 \rtimes \mathfrak{su}(1,1), \tag{2.4}$$

where $\mathfrak{h}_1$ is an ideal in $\mathfrak{g}$ determined by the commutation relations:

$$[a, a^+] = 1, \tag{2.5a}$$

$$\left[K_0, K_\pm\right] = \pm K_\pm, \quad \left[K_-, K_+\right] = 2K_0, \tag{2.5b}$$

$$\left[a, K_+\right] = a^+, \quad \left[K_-, a^+\right] = a, \tag{2.5c}$$

$$\left[K_+, a^+\right] = \left[K_-, a\right] = 0, \tag{2.5d}$$

$$\left[K_0, a^+\right] = \frac{1}{2}a^+, \quad \left[K_0, a\right] = -\frac{1}{2}a. \tag{2.5e}$$

We associate to the generators a, a^+ of the HW group and to the generators $K_{0,+,-}$ of the group $SU(1, 1)$ the operators a, a^+, respectively $\boldsymbol{K}_{0,+,-}$, where $(a^+)^+ = a$, $\boldsymbol{K}_0^+ = \boldsymbol{K}_0$, $\boldsymbol{K}_\pm^+ = \boldsymbol{K}_\mp$. We impose to the cyclic vector e_0 the conditions

$$ae_0 = 0, \tag{2.6a}$$

$$\boldsymbol{K}_- e_0 = 0, \tag{2.6b}$$

$$\boldsymbol{K}_0 e_0 = k e_0; \quad k > 0, \; 2k = 2, 3, \ldots. \tag{2.6c}$$

In (2.6c) we consider the positive discrete series representations D_k^+ of $SU(1, 1)$ (cf. [1]).

We define Perelomov's coherent state vectors [18]

$$e_{z,w} := e^{za^+ + w\boldsymbol{K}_+} e_0, \quad z \in \mathbb{C}, \quad |w| < 1. \tag{2.7}$$

based on the manifold

$$M := HW/\mathbb{R} \times SU(1,1)/U(1), \tag{2.8a}$$

$$M = \mathcal{D} := \mathbb{C} \times \mathcal{D}_1. \tag{2.8b}$$

The general scheme [7, 8] associates to elements of the Lie algebra $\mathfrak{g}$ first order holomorphic differential operators $X \in \mathfrak{g} \rightarrow \mathbb{X}$ with polynomial coefficients.

Lemma 1. *The differential action of the generators* (2.5a)–(2.5e) *of the Jacobi algebra* (2.4) *is given by the formulas:*

$$\boldsymbol{a} = \frac{\partial}{\partial z}; \quad \boldsymbol{a}^+ = z + w\frac{\partial}{\partial z}; \tag{2.9a}$$

$$\mathbb{K}_- = \frac{\partial}{\partial w}; \quad \mathbb{K}_0 = k + \frac{1}{2}z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}; \tag{2.9b}$$

$$\mathbb{K}_+ = \frac{1}{2}z^2 + 2kw + zw\frac{\partial}{\partial z} + w^2\frac{\partial}{\partial w}. \tag{2.9c}$$

Sketch of the proof. We start with the formal relations:

$$a^+ e_{z,w} = \frac{\partial}{\partial z} e_{z,w}; \quad \boldsymbol{K}_+ e_{z,w} = \frac{\partial}{\partial w} e_{z,w}.$$

The proof is based on the general formula

$$\mathrm{Ad}(\exp X) = \exp(\mathrm{ad}_X), \tag{2.10}$$

valid for Lie algebras $\mathfrak{g}$, which here we write down explicitly as

$$Ae^X = e^X\left(A - [X, A] + \frac{1}{2}[X, [X, A]] + \ldots\right), \tag{2.11}$$

and we take $X = za^+ + w\boldsymbol{K}_+$ because of the definition (2.7).

3. THE REPRODUCING KERNEL

Lemma 2. Let $K = K(\bar{z}, \bar{w}, z, w)$, *where* $z \in \mathbb{C}$, $w \in \mathbb{C}$, $|w| < 1$,

$$K := (e_0, e^{\bar{z}a + \bar{w}\boldsymbol{K}_-} e^{za^+ + w\boldsymbol{K}_+} e_0). \tag{3.1}$$

Then the reproducing kernel is

$$K = (1 - w\bar{w})^{-2k} \exp\frac{2z\bar{z} + z^2\bar{w} + \bar{z}^2 w}{2(1 - w\bar{w})}. \tag{3.2}$$

More generally, the kernel $K : M \times \bar{M} \to \mathbb{C}$ *is:*

$$K(z, w; \bar{z}', \bar{w}') := (e_{\bar{z},\bar{w}}, e_{\bar{z}',\bar{w}'}) = (1 - w\bar{w}')^{-2k} \exp\frac{2\bar{z}'z + z^2\bar{w}' + \bar{z}'^2 w}{2(1 - w\bar{w}')}. \tag{3.3}$$

Sketch of the proof. We introduce the auxiliary operators:

$$\boldsymbol{K}_+ = \frac{1}{2}(a^+)^2 + \boldsymbol{K}'_+, \tag{3.4a}$$

$$\boldsymbol{K}_- = \frac{1}{2}a^2 + \boldsymbol{K}'_-, \tag{3.4b}$$

$$\boldsymbol{K}_0 = \frac{1}{2}\left(a^+ a + \frac{1}{2}\right) + \boldsymbol{K}'_0. \tag{3.4c}$$

We use the relation

$$(e_0, e^{\overline{w}\boldsymbol{K}'_-} e^{w'\boldsymbol{K}'_+} e_0) = (1 - w'\overline{w})^{-2k'}, \quad k = k' + \frac{1}{4}. \tag{3.5}$$

Recall (cf. [1]) that $e_{k,k+m}$ *is an orthonormal system*, where

$$e_{k,k+m} := a_{km}(\boldsymbol{K}_+)^m e_{k,k}; \quad a_{km}^2 = \frac{\Gamma(2k)}{m!\Gamma(m+2k)}. \tag{3.6}$$

We use also the orthonormality of the n-particle states:

$$| n > = (n!)^{-\frac{1}{2}} (a^+)^n \,| 0 >; \; < n', n > = \delta_{nn'}. \tag{3.7}$$

We introduce a notation $E = E(z, w)$ and we write down

$$\begin{aligned} E(z, w) &:= e^{za^+ + \frac{w}{2}(a^+)^2} = \sum_{p,q\geq 0} \frac{z^p}{p!} \frac{\left(\frac{w}{2}\right)^q}{q!} (a^+)^{p+2q} = \\ &= \sum_{n\geq 0} \sum_{q=0}^{\left[\frac{n}{2}\right]} \frac{z^{n-2q}}{(n-2q)!q!} \left(\frac{w}{2}\right)^q (a^+)^n. \end{aligned} \tag{3.8}$$

We use the following relations of the Hermite polynomials (cf. 10.13.9 and 10.13.22 in [4]):

$$H_n(x) = n! \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^m (2x)^{n-2m}}{m!(n-2m)!}, \tag{3.9}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{s}{2}\right)^n}{n!} H_n(x) H_n(y) = \frac{1}{\sqrt{1-s^2}} \exp \frac{2xys - (x^2+y^2)s^2}{1-s^2}, \quad | s | < 1. \tag{3.10}$$

4. THE GROUP ACTION

Let us recall some relations for the displacement operator (see *e.g.*, [13]):

$$D(\alpha) := \exp(\alpha a^+ - \overline{\alpha} a) = \exp\left(-\frac{1}{2} | \alpha |^2\right) \exp(\alpha a^+) \exp(-\overline{\alpha} a), \tag{4.1}$$

$$D(\alpha_2)D(\alpha_1)=e^{i\theta_h(\alpha_2,\alpha_1)}D(\alpha_2+\alpha_1),\quad \theta_h(\alpha_2,\alpha_1):=\Im(\alpha_2\bar{\alpha}_1). \tag{4.2}$$

We denote by S the D_+^k representation of the group $SU(1, 1)$ and let us introduce the notation $\underline{S}(z)=S(w)$. We have (see *e.g.*, [18]):

$$\underline{S}(z):=\exp(z\boldsymbol{K}_+-\bar{z}\boldsymbol{K}_-),\quad z\in\mathbb{C}; \tag{4.3a}$$

$$S(w)=\exp(w\boldsymbol{K}_+)\exp(\eta\boldsymbol{K}_0)\exp(-\bar{w}\boldsymbol{K}_-); \tag{4.3b}$$

$$w=w(z)=\frac{z}{|z|}\tanh(|z|),\quad w\in C,\quad |w|<1; \tag{4.3c}$$

$$z=z(w)=\frac{w}{|w|}\operatorname{arctanh}(|w|)=\frac{w}{2|w|}\log\frac{1+|w|}{1-|w|}; \tag{4.3d}$$

$$\eta=\log(1-w\bar{w})=-2\log(\cosh(|z|)). \tag{4.3e}$$

Let us consider an element $g\in SU(1,1)$,

$$g=\begin{pmatrix} a & b\\ \bar{b} & \bar{a}\end{pmatrix},\quad \text{where } |a|^2-|b|^2=1. \tag{4.4}$$

Remark 1. 1. *The following relations hold:*

$$\underline{S}(z)e_0=(1-|w|^2)^k e_{0,w}, \tag{4.5}$$

$$e_g:=S(g)e_0=\bar{a}^{-2k}e_{0,w=-i\frac{b}{\bar{a}}}=\left(\frac{a}{\bar{a}}\right)^k\underline{S}(z)e_0, \tag{4.6}$$

$$S(g)e_{0,w}=(\bar{a}+\bar{b}w)^{-2k}e_{0,g\cdot w}. \tag{4.7}$$

The action of the group $SU(1, 1)$ *on the unit disk* $\mathcal{D}_1:=SU(1,1)/U(1)$ *in* (4.7) *is*

$$g\cdot w=\frac{aw+b}{\bar{b}w+\bar{a}}. \tag{4.8}$$

2. *If* $\underline{S}(z)$ *is defined by* (4.3a), *then* (cf. [10]):

$$\underline{S}(z_2)\underline{S}(z_1)=\underline{S}(z_3)e^{i\theta_s\boldsymbol{K}_0}; \tag{4.9a}$$

$$w_3=\frac{w_1+w_2}{1+\bar{w}_2w_1}; \tag{4.9b}$$

$$e^{i\theta_s}=\frac{1+w_2\bar{w}_1}{1+w_1\bar{w}_2}. \tag{4.9c}$$

In equation (4.9b) w_i *and* z_i, i = 1, 2, 3, *are related by the relations* (4.3c), (4.3d).

We recall the so called (see *e.g.*, [17]) *Holstein-Primakoff-Bogoliubov equations*:

$$\underline{S}^{-1}(z)\,a\,\underline{S}(z)=\cosh(|z|)\,a+\frac{z}{|z|}\sinh(|z|)\,a^{+}, \tag{4.10a}$$

$$\underline{S}^{-1}(z)\,a^{+}\,\underline{S}(z)=\cosh(|z|)\,a^{+}+\frac{\overline{z}}{|z|}\sinh(|z|)\,a. \tag{4.10b}$$

Let us introduce the notation:

$$\tilde{A}:=\begin{pmatrix} A \\ \overline{A} \end{pmatrix};\quad \mathcal{D}=\mathcal{D}(z)=\begin{pmatrix} M & N \\ P & Q \end{pmatrix}, \tag{4.11}$$

$$M=\cosh(|z|);\quad N=\frac{z}{|z|}\sinh(|z|);\quad P=\overline{N};\quad Q=M. \tag{4.12}$$

We have

$$\mathcal{D}(z)=e^{X},\quad \text{where}\quad X:=\begin{pmatrix} 0 & z \\ \overline{z} & 0 \end{pmatrix}. \tag{4.13}$$

Remark 2. *In the notation* (4.11), (4.12), *equations* (4.10) *become:*

$$\underline{S}^{-1}(z)\tilde{a}\underline{S}(z)=\mathcal{D}(z)\tilde{a}.$$

Remark 3. 1. *If D and $\underline{S}(z)$ are defined by* (4.1), *respectively* (4.3a), *then*

$$D(\alpha)\underline{S}(z)=\underline{S}(z)D(\beta), \tag{4.14}$$

$$\tilde{\beta}=\mathcal{D}(-z)\tilde{\alpha};\quad \tilde{\alpha}=\mathcal{D}(z)\tilde{\beta}. \tag{4.15}$$

2. *If*

$$\underline{S}(z,\theta):=\exp(2i\theta\boldsymbol{K}_0+z\boldsymbol{K}_{+}-\overline{z}\boldsymbol{K}_{-}). \tag{4.16}$$

then

$$\underline{S}(z,\theta)^{-1}(z)\,a\,\underline{S}(z,\theta)=(\mathrm{cs}(x)+i\theta\frac{\mathrm{si}(x)}{x})\,a+z\frac{\mathrm{si}(x)}{x}a^{+}, \tag{4.17a}$$

$$\underline{S}(z,\theta)^{-1}(z)\,a^{+}\,\underline{S}(z,\theta)=(\mathrm{cs}(x)-i\theta\frac{\mathrm{si}(x)}{x})\,a^{+}+\overline{z}\frac{\mathrm{si}(x)}{x}a, \tag{4.17b}$$

where

$$\mathrm{cs}(x):=\begin{cases}\cosh(x), & \text{if } \lambda=x^2>0,\\ \cos(x), & \text{if } \lambda=-x^2<0,\end{cases}\qquad \lambda:=|z|^2-\theta^2, \tag{4.18}$$

and similarly for $\mathrm{si}(x)$.

If $X \in \mathfrak{su}(1,1)$,

$$X = \begin{pmatrix} i\theta & z \\ \bar{z} & -i\theta \end{pmatrix}, \quad \theta \in \mathbb{R}, \quad z \in \mathbb{C}, \tag{4.19}$$

then $g = e^X \in SU(1,1)$ is an element of the form (4.4), where

$$a = \mathrm{cs}(x) + i\theta \frac{\mathrm{si}(x)}{x}, \quad b = z\frac{\mathrm{si}(x)}{x}. \tag{4.20}$$

If $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(1,1)$, then equations (4.17) can be written down as

Remark 4. *If* S *denotes the representation of* $SU(1,1)$ *and* $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(1,1)$, *then*

$$S^{-1}(g)\tilde{a}S(g) = g \cdot \tilde{a}. \tag{4.21}$$

Remark 5. *In the matrix realization* (2.3), *we have*

$$S(g)D(\alpha)S^{-1}(g) = D(\alpha_g), \tag{4.22}$$

where the natural action of $SU(1,1)\times\mathbb{C}\rightarrow\mathbb{C}$: $g\cdot\tilde{\alpha} := \alpha_g$ *is*

$$\alpha_g = a\alpha + b\bar{\alpha}, \tag{4.23}$$

and a, b have the expression (4.20).

Remark 6. *The action:* $(\alpha_2, z_2)\times(\alpha_1, w_1) = (A, w)$, *where* z_2, $\alpha_{1,2}$, $A \in \mathbb{C}$, w, $w_1 \in \mathcal{D}_1$ *and the variables of type w and z are related by equations* (4.3c), (4.3d), *can be expressed as:*

$$A = \alpha_2 + \alpha_1 \cosh|z_2| + \bar{\alpha}_1 \frac{z_2}{|z_2|}\sinh|z_2| = \alpha_2 + \frac{\alpha_1 + \bar{\alpha}_1 w_2}{(1-|w_2|^2)^{1/2}}, \tag{4.24a}$$

$$w = \frac{\cosh|z_2|w_1 + \frac{z_2}{|z_2|}\sinh|z_2|}{\frac{\bar{z}_2}{|z_2|}\sinh|z_2|w_1 + \cosh|z_2|} = \frac{w_1 + w_2}{1 + w_1\bar{w}_2}. \tag{4.24b}$$

Equations (4.24) *express the action* $(\alpha_2, w_2)\times(\alpha_1, w_1) = (\alpha_2 + w_2 \circ \alpha_1, w_2 \circ w_1)$, $\alpha_{1,2} \in \mathbb{C}$, $w_{1,2} \in \mathcal{D}_1$. (4.24) *can be written down as:*

$$\tilde{A} = \tilde{\alpha}_2 + \mathcal{D}\tilde{\alpha}_1, \tag{4.25a}$$

$$w = \frac{Mw_1 + N}{Pw_1 + Q}. \tag{4.25b}$$

Let us introduce the ("squeezed") normalized vectors:

$$\Psi_{\alpha,w} := D(\alpha)S(w)e_0; \quad \alpha \in \mathbb{C}, \quad w \in \mathbb{C}, \quad |w| < 1. \tag{4.26}$$

Remark 7. *The product of the representations* D *and* $\underline{S}$ *acts on the CS-vector* (4.26) *with the effect:*

$$D(\alpha_2)\underline{S}(z_2)\Psi_{\alpha_1,w_1} = J\Psi_{A,w}, \quad \textit{where} \quad J = e^{i(\theta_h(\alpha_2,\alpha)+k\theta_s)}. \tag{4.27}$$

(A, w) *are given by* Remark 6, $\theta_h(\alpha_2,\alpha)$ *is given by* (4.2) *with* α *given by* (4.28) *and*

$$\tilde{\alpha} = \mathcal{D}(z)\tilde{\alpha}_1. \tag{4.28}$$

Note also that the Remark 7 has an important consequence well known in the quantum optics of squeezed states (see *e.g.* equation (20) p. 3219 in [19]).

Lemma 3. *The vectors* (4.26), (2.7), *i.e.*

$$\Psi_{\alpha,w} := D(\alpha)S(w)e_0; \quad e_{z,w'} := \exp(za^+ + w'\boldsymbol{K}_+)e_0,$$

are related by the relation

$$\Psi_{\alpha,w} = (1 - w\overline{w})^k \exp\left(-\frac{\overline{\alpha}}{2}z\right)e_{z,w}, \tag{4.29}$$

where $z = \alpha - w\overline{\alpha}$.

For the proof, we use the relations (4.3a), (4.3b), (2.6b), (2.6c), and (2.10) where $Z = -\overline{\alpha}a$; $X = w\boldsymbol{K}_+$.

Proposition 1. *Let us consider the action* $S(g)D(\alpha)e_{z,w}$, *where* $g \in SU(1,1)$ *has the form* (4.4), $D(\alpha)$ *is given by* (4.1), *and the coherent state vector is defined in* (2.7). *Then we have the formula* (4.30) *and the relations* (4.31), (4.32)–(4.34):

$$S(g)D(\alpha)e_{z,w} = \lambda e_{z_1,w_1}, \quad \lambda = \lambda(g, \alpha; z, w), \tag{4.30}$$

$$z_1 = \frac{\alpha - \overline{\alpha}w + z}{\overline{b}w + \overline{a}}; \quad w_1 = g \cdot w = \frac{aw + b}{\overline{b}w + \overline{a}}, \tag{4.31}$$

$$\lambda = (\overline{a} + \overline{b}w)^{-2k} \exp\left(\frac{z}{2}\overline{\alpha}_0 - \frac{z_1}{2}\overline{\alpha}_2\right)\exp i\theta_h(\alpha, \alpha_0), \tag{4.32}$$

$$\alpha_0 = \frac{z + \overline{z}w}{1 - w\overline{w}}, \tag{4.33}$$

$$\alpha_2 = (\alpha + \alpha_0)a + (\bar{\alpha} + \bar{\alpha}_0)b. \tag{4.34}$$

Corollary 1. *The action of the Jacobi group*

$$G := HW \rtimes SU(1,1), \tag{4.35}$$

on the manifold (2.8) *is given by equations* (4.30), (4.31). *The composition law in G is*

$$(g_1, \alpha_1, t_1) \circ (g_2, \alpha_2, t_2) = (g_1 \circ g_2, g_2^{-1} \cdot \alpha_1 + \alpha_2, t_1 + t_2 + \Im(g_2^{-1} \cdot \alpha_1 \bar{\alpha}_2)), \tag{4.36}$$

where $g \cdot \alpha := \alpha_g$ *is given by* (4.23), *and if g has the form given by* (4.4), *then* $g^{-1} \cdot \alpha = \bar{a}\alpha - b\bar{\alpha}$.

The proof of Proposition 1 is based on the Remarks of this section and on Lemma 3.

5. THE HILBERT SPACE OF HOLOMORPHIC FUNCTIONS

If $\varphi : M \to \mathcal{H}^*$ is Perlomov's CS-map, we recall [7, 8] the construction of the map

$$\Phi : \mathcal{H}^\star \to \mathcal{F}_H, \quad \Phi(\psi) := f_\psi, \quad f_\psi(z) = \Phi(\psi)(z) = (\varphi(z), \psi)_{\mathcal{H}} = (e_{\bar{z}}, \psi)_{\mathcal{H}}, \tag{5.1}$$

and the isometric embedding

$$(\psi_1, \psi_2)_{\mathcal{H}^*} = (\Phi(\psi_1), \Phi(\psi_2))_{\mathcal{F}_\mathcal{H}} = (f_{\psi_1}, f_{\psi_2})_{\mathcal{F}_\mathcal{H}} = \int_M \bar{f}_{\psi_1}(z) f_{\psi_2}(z) \mathrm{d}\nu_M(z). \tag{5.2}$$

Perelomov's CS-vectors associated to the *HW* group are (see *e.g.*, [13])

$$e_z := e^{za^+} e_0 = \sum \frac{z^n}{(n!)^{1/2}} | n >, \tag{5.3}$$

and their corresponding functions are (see *e.g.*, [2])

$$f_{|n>}(z) := (e_{\bar{z}}, | n >) = \frac{z^n}{(n!)^{1/2}}. \tag{5.4}$$

The reproducing kernel $K : \mathbb{C} \times \bar{\mathbb{C}} \to \mathbb{C}$ is

$$K(z, \bar{z}') := (e_{\bar{z}}, e_{\bar{z}'}) = \sum f_{|n>}(z) \bar{f}_{|n>}(z') = e^{z\bar{z}'}. \tag{5.5}$$

The scalar product on the Segal-Bargmann-Fock space is (cf. [2])

$$(\phi, \psi)_{\mathcal{H}^*} = (f_\phi, f_\psi)_{\mathcal{F}_\mathcal{H}} = \frac{1}{\pi} \int \bar{f}_\phi(z) f_\psi(z) e^{-|z|^2} \mathrm{d}\Re z \, \mathrm{d}\Im z.$$

Perelomov's CS-vectors for $SU(1, 1)$ based on the manifold $\mathcal{D}_1 = SU(1,1)/U(1)$ are

$$e_z := e^{z\boldsymbol{K}_+} e_0 = \sum \frac{z^n \mathbf{K}_+^n}{n!} e_0 = \sum \frac{z^n e_{k,k+n}}{n! a_{kn}}, \tag{5.6}$$

and the corresponding functions are (see *e.g.*, [1])

$$f_{e_{k,k+n}}(z) := (e_{\bar{z}}, e_{k,k+n}) = \sqrt{\frac{\Gamma(n+2k)}{n!\Gamma(2k)}}\, z^n. \tag{5.7}$$

The reproducing kernel $K : \mathcal{D}_1 \times \bar{\mathcal{D}}_1 \to \mathbb{C}$ is

$$K(z, \bar{z}') := (e_{\bar{z}}, e_{\bar{z}'}) = \sum f_{e_{k,k+m}}(z) \bar{f}_{e_{k,k+m}}(z') = (1 - z\bar{z}')^{-2k}. \tag{5.8}$$

The scalar product on $\mathcal{D}_1$ is (see *e.g.*, [1])

$$(\phi, \psi)_{\mathcal{H}^*} = (f_\phi, f_\psi)_{\mathcal{F}_\mathcal{H}} = \frac{2k-1}{\pi} \int_{|z|<1} \bar{f}_\phi(z) f_\psi(z) (1 - |z|^2)^{2k-2} \mathrm{d}\Re z \, \mathrm{d}\Im z, \;\; 2k = 2, 3, \ldots.$$

In formula (2.7) defining Perelomov's CS vectors for the Jacobi group (4.35), we have

$$e_{z,w} = \exp\left(z a^+ + \frac{1}{2}(a+)^2 w \right) \exp(w \boldsymbol{K}'_+) e_0.$$

With (3.8) and (3.9), we have

$$e_{z,w} = \sum_n \frac{i^{-n}}{n!} \left(\frac{w}{2} \right)^{\frac{n}{2}} H_n\left(\frac{iz}{\sqrt{2w}} \right) (a^+)^n \sum_m \frac{w^m}{m!} (\boldsymbol{K}'_+)^m e_0.$$

Now we take into account (3.6) and we get

$$e_{z,w} = \sum_n \frac{i^{-n}}{(n!)^{1/2}} | n > \left(\frac{w}{2} \right)^n H_n\left(\frac{iz}{\sqrt{2w}} \right) \sum_m \frac{w^m}{m! a_{k'm}} e_{k',k'+m}.$$

The base of functions associated to the CS-vectors attached to the Jacobi group (4.35), based on the manifold M (2.8), are

$$f_{|n>; e_{k',k'+m}}(z, w) = f_{e_{k',k'+m}}(w) \frac{P_n(z, w)}{\sqrt{n!}}, \quad z \in \mathbb{C}, \quad |w| < 1, \tag{5.9}$$

where the functions $f_{e_{k',k'+m}}$ are defined in (5.7) and, with formula (3.9), we obtain

$$P_n(z,w)=n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\left(\frac{w}{2}\right)^k\frac{z^{n-2k}}{k!(n-2k)!}. \tag{5.10}$$

The first 6 polynomials $P_n(z,w)$ are

$$\begin{aligned} &P_0=1; && P_1=z;\\ &P_2=z^2+w; && P_3=z^3+3zw;\\ &P_4=z^4+6z^2w+3w^2; && P_5=z^5+10z^3w+15zw. \end{aligned} \tag{5.11}$$

The reproducing kernel (3.3) $K:M\times\bar{M}\to\mathbb{C}$ has the property:

$$K(z,w;\bar{z},\bar{w}'):=(e_{\bar{z},\bar{w}},e_{\bar{z}',\bar{w}'})=\sum_{n,m}f_{|n>,e_{k',k'+m}}(z,w)\bar{f}_{|n>,e_{k',k'+m}}(z',w')= \tag{5.12a}$$

$$=(1-w\bar{w}')^{-2k}\exp\frac{2\bar{z}'z+z^2\bar{w}'+\bar{z}'^2w}{2(1-w\bar{w}')}. \tag{5.12b}$$

The scalar product of functions from the space $\mathfrak{F}_K$ corresponding to the kernel defined by (3.3) on the manifold (2.8b) is:

$$(\phi,\psi)=$$
$$=\Lambda\int_{z\in\mathbb{C};|w|<1}\bar{f}_\phi(z,w)f_\psi(z,w)(1-w\bar{w})^{2k}\exp-\frac{|z|^2}{1-w\bar{w}}\exp-\frac{z^2\bar{w}+\bar{z}^2w}{2(1-w\bar{w})}\mathrm{d}\nu, \tag{5.13}$$

where the value of the G-invariant measure $\mathrm{d}\nu$

$$\mathrm{d}\nu=\frac{\mathrm{d}\Re w\,\mathrm{d}\Im w}{(1-w\bar{w})^3}\mathrm{d}\Re z\,\mathrm{d}\Im z \tag{5.14}$$

will be deduced latter in (5.20).

In order to find the value of the constant Λ in (5.13), we take the functions $\phi,\ \psi=1$, we change the variable $z\to(1-w\bar{w})^{1/2}z$ and, applying equations (A1), (A2) in [3], we get

$$\Lambda=\frac{4k-3}{2\pi^2}. \tag{5.15}$$

Now we calculate the Kähler potential as the logarithm of the reproducing kernel (3.3), $f:=\log K$,

$$f=\frac{2z\bar{z}+z^2\bar{w}+\bar{z}^2w}{2(1-w\bar{w})}-2k\log(1-w\bar{w}). \tag{5.16}$$

The Kähler two-form ω is given by the formula:

$$-i\omega = f_{z\bar{z}}\mathrm{d}z \wedge \mathrm{d}\bar{z} + f_{z\bar{w}}\mathrm{d}z \wedge \mathrm{d}\bar{w} + f_{\bar{z}w}\mathrm{d}\bar{z} \wedge \mathrm{d}w + f_{w\bar{w}}\mathrm{d}w \wedge \mathrm{d}\bar{w}. \tag{5.17}$$

The volume form is:

$$\omega \wedge \omega = -2\begin{vmatrix} f_{z\bar{z}} & f_{z\bar{w}} \\ f_{\bar{z}w} & f_{\bar{w}w} \end{vmatrix}\mathrm{d}z \wedge \mathrm{d}\bar{z} \wedge \mathrm{d}w \wedge \mathrm{d}\bar{w}. \tag{5.18}$$

We find for the manifold (2.8) *the fundamental two-form* ω (5.17), *where*

$$f_{z\bar{z}} = \frac{1}{1 - w\bar{w}}, \tag{5.19a}$$

$$f_{z\bar{w}} = \frac{z + w\bar{z}}{(1 - w\bar{w})^2}, \tag{5.19b}$$

$$f_{w\bar{w}} = \frac{(\bar{z} + \bar{w}z)(z + w\bar{z})}{(1 - w\bar{w})^3} + \frac{2k}{(1 - w\bar{w})^2}. \tag{5.19c}$$

For the volume form (5.18), we find:

$$\omega \wedge \omega = 4k(1 - w\bar{w})^{-3} 4\Re z \Im z \Re w \Im w. \tag{5.20}$$

It can be checked up that indeed, *the measure* $\mathrm{d}\nu$ *and the fundamental two-form* ω *are group-invariant at the action* (4.31) *of the Jacobi group* (4.35).

Now we summarize the contents of this section as follows:

Proposition 2. *Let us consider the Jacobi group G* (4.35) *with the composition rule* (4.36) *acting on the coherent state manifold* (2.8) *via equation* (4.31). *The manifold* $\mathcal{D}$ *has the Kähler potential* (5.16) *and the G-invariant Kähler two-form* ω *given by* (5.17), (5.19). *The holomorphic polynomials associated to the coherent state vectors* (2.7) *are given by* (5.9), *where the functions f are given by* (5.7), *while the polynomials P are given by* (5.10). *The Hilbert space of holomorphic functions* $\mathcal{F}_K$ *associated to the holomorphic kernel* $K : M \times \bar{M} \to \mathbb{C}$ *given by* (3.3) *is endowed with the scalar product* (5.13), *where the normalization constant* Λ *is given by* (5.15) *and the G-invariant measure* $\mathrm{d}\nu$ *is given by* (5.14).

Proposition 3. *Let* $h := (g, \alpha) \in G$, *where G is the Jacobi group* (4.35), *and we consider the representation* $\pi(h) := S(g)D(\alpha)$, $g \in SU(1,1)$, $\alpha \in \mathbb{C}$, *and let the notation* $x := (z, w) \in \mathcal{D} := \mathbb{C} \times \mathcal{D}_1$. *Then the continuous unitary representation* $(\pi_K, \mathcal{H}_K)$ *attached to the positive definite holomorphic kernel K defined by* (3.3) *is*

$$(\pi_K(h).f)(x) = J(h^{-1}, x)^{-1} f(h^{-1}.x), \tag{5.21}$$

where the cocycle $J(h^{-1}, x)^{-1} := \lambda(h^{-1}, x)$ *with* λ *defined by equations* (4.30)–

(4.34) *and the function f belongs to the Hilbert space of holomorphic functions* $\mathcal{H}_K \equiv \mathcal{F}_K$ *endowed with the scalar product* (5.13), *where* Λ *is given by* (5.15).

Remark 8. *The motion on the manifold* (2.8) *generated by the linear Hamiltonian*

$$\boldsymbol{H} = \epsilon_a a + \overline{\epsilon}_a a^+ + \epsilon_0 \boldsymbol{K}_0 + \epsilon_+ \boldsymbol{K}_+ + \epsilon_- \boldsymbol{K}_-. \tag{5.22}$$

is governed by the matrix Riccati equation:

$$i\dot{z} = \epsilon_a + \frac{\epsilon_0}{2} z + \epsilon_+ z w, \tag{5.23a}$$

$$i\dot{w} = \epsilon_- + (\overline{\epsilon}_a + \epsilon_0) w + \epsilon_+ w^2. \tag{5.23b}$$

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